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A NEW PROBLEM OF HYDRODYNAMIC LUBRICATION WITH TEMPERATURE AND VISCOSITY VARIATIONS IN GAP HEIGHT DIRECTION

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Abstract

Numerous Authors of scientific papers occurring in hydrodynamic theory of slide bearing lubrication, up to now almost always had neglected the oil dynamic viscosity variations caused by temperature, adhesion forces, capillary forces, or hydrogen ion concentration across the film thickness by virtue of the statement of the constant temperature as well constant adhesion as capillary forces in the thin bearing gap height direction. In addition, simultaneously by virtue of boundary layer simplifications for energy equation and by virtue of new measurements performed in micro and nano- level follows that oil temperature gradients and its values differences and adhesion or capillary gradients of forces in bearing gap height directions are not negligible small. The contemporary hydrodynamic theory of lubrication for non-isothermal lubricant flow is unfortunately based on the assumption of constant viscosity values across the film thickness, despite abovementioned self-evident contradiction between the fact of constant viscosity and simultaneously temperature variations in gap height direction. Such problem was up to now not sufficient critical examined and explained in practical and theoretical sense. After Authors, knowledge by virtue of above problem the most scientific papers in the domain of non-isothermal and lamellar hydrodynamic slide bearing or biobearing lubrication were up to now not sufficient correctly solved. It denotes that the main hydrodynamic lubrication solutions presenting for example hydrodynamic pressure by the modified Reynolds equation and temperature by the energy conservation equation are not sufficiently correctly obtained and next not correctly solved. Therefore are assumed simultaneously the temperature T and oil dynamic viscosity variations in length, width and bearing gap- height directions. From this assumption follows, that the energy equation must be solved simultaneously with the equations of motion i.e. consequently with pressure equation where viscosity depends on temperature and temperature depends on the coordinate in gap height direction.

Keywords: *liquid viscosity changes in thin gap height direction, temperature, adhesion or capillary forces variations in thin gap height direction, new analytical solutions*

1. Introduction

This paper presents a new idea of solution of the Newtonian, unsymmetrical, stationary, non-isothermal, time independent slide bearing lubrication in orthogonal, curvilinear (α_1, α_2, α_3) coordinates for lubricant viscosity and temperature variations in gap height direction. Liquid inertia forces and terms of energy convections are neglected, pressure p is constant in gap height direction and bearing rotational, surfaces have non-monotone generating lines.

Up to here in scientific papers describing non-isothermal hydrodynamic lubrication problems, many Authors almost always had been neglected the lubricant viscosity variations caused by the temperature in ultra-thin gap height direction or are neglected viscosity variations caused by the hydrogen ion concentration as well adhesion as capillary forces variations across the gap height and simultaneously consequently had been required assumption that temperature, adhesion or capillary forces are constant in gap height direction [1-8]. Such assumption was taken into account by virtue of the statement of very small temperature, adhesion or capillary forces differences in bearing or biobearing ultra-thin gap height direction. In reality from the form of energy equation obtained by virtue of boundary layer simplifications, and by virtue of a new AFM measurements performed in nano-level and on the ground of the new chemical achievements referring the dissociated hydrogen ion concentration follows, that temperature changes, hydrogen ion concentrations intensity, capillary forces not only in circumference and bearing or biobearing length direction are significant, but also especially the temperature gradients variations, dissipated hydrogen ion concentration, adhesion or capillary forces across the film thickness in radial direction are very relative large [5, 6, 9, 10].

Up to here in non-isothermal lubrication problems, the Author and research team in own foreseen papers are indeed taken into account the temperature variations in gap height directions accordingly with the energy equations, but the oil viscosity variations caused by temperature across the film thickness during the integration of the momentum conservation equations have been assumed not in first but in second or further steps of resolved approximation forms of the lubrication problem [11-13].

After Authors suggestion both abovementioned methods are not sufficient correctly considered in aspects of a new contemporary applications dealing the hydrodynamic bearing lubrication problems.

Hence in this paper are assumed the oil dynamic viscosity variations in length α_1, width α_3 and gap- height directions α_2, simultaneously with the variations across the film thickness α_2 for temperature T, adhesion or capillary forces, hydrogen ion concentrations intensities changes. For example temperature changes the dynamic viscosity values namely decreases the viscosity values with the temperature increments. Moreover capillary forces, adhesion forces and hydrogen ion concentration forces increments have simultaneously influence on the lubricant dynamic viscosity extension and enhancement of values near the two cooperating surfaces limiting the thin layer liquid flow in bearing or biobearing gap.

After Author, knowledge presented in this paper model is more correct and reasonable in comparison with up to now applied methods mostly for very small gap heights. In such cases for example in micro-bearing gaps, admittedly the temperature differences and differences of oil dynamic viscosity values depended on temperature in the region between sleeve and journal surface are relative small, but gradients of temperature, as well as gradients in gap height direction of oil dynamic viscosity values caused by the temperature or adhesion as well as capillary forces are significantly large. Description of this problem requires the consideration of equation of motion (i.e. modified Reynolds equation) with compactly conjugated energy conservation equation in non-linear form. The solutions of the set of two above mutually dependent partial differential equations tend to the solutions of nonlinear partial differential and integral equations. Such model illustrates both effects namely mutually influences of temperature on the hydrodynamic pressure and the influences of temperature on the pressure.

However, we can observe that for constant dynamic viscosity obtained from assumption for independent viscosity on temperature, the above derived differential equations of modified Reynolds and Energy equations are quasi-independent where pressure function is independent on temperature but temperature depends on pressure because viscosity function changes additionally in circumferential and length bearing directions. Authors hope that the revision of calculations enables to show the contemporary analytical solutions of journal bearing for non-isothermal lubrication as well for Newtonian as for non-Newtonian lubricants in form that is more realistic.

2. Ideas of a new general non-isothermal lubrication problem formulation

We assume the dimensional total oil dynamic viscosity in the following form [12]:

$$\eta_\Sigma(\alpha_1,\alpha_2,\alpha_3)=\eta_0\,\eta_1(\alpha_1,\alpha_3)\,\eta_\pi(\alpha_1,\alpha_2,\alpha_3),$$
$$\eta_\pi(\alpha_1,\alpha_2,\alpha_3)\equiv\eta_\mathrm{T}(\alpha_1,\alpha_2,\alpha_3)\,\eta_\mathrm{H}(\alpha_1,\alpha_2,\alpha_3)\,\eta_\mathrm{A}(\alpha_1,\alpha_2,\alpha_3)\,\eta_\mathrm{C}(\alpha_1,\alpha_2,\alpha_3), \tag{1}$$
$$\eta_\mathrm{T}(\alpha_1,\alpha_2,\alpha_3)\equiv \mathrm{e}^{-\delta\cdot T(\alpha_1,\alpha_2,\alpha_3)},$$

where: η_Σ – total dynamic viscosity, η_π – product of influences of viscosities η_T, η_H, η_A, η_C caused by the Temperature. Hydrogen ion concentration consistency, Adhesion and Capillary forces, respectively. Constant dimensional coefficient δ in [K^{-1}] describes the influences of dimensional temperature T on the oil dynamic viscosity.

These abovementioned influences are functions of three orthogonal co-ordinates (α_1, α_2, α_3,). Dimensional dynamic viscosity functions of the lubricant no provoked by the temperature, adhesion forces, capillary forces, power hydrogen ion concentration pH, but provoked by the many other influences for example among other by the pressure, are independent on gap height coordinate α_2 and are denoted by $\eta=\eta_0\eta_1$, where $\eta_1(\alpha_1,\alpha_3)$ – dimensionless oil dynamic viscosity, η_0 – characteristic dimensional oil dynamic viscosity value. Putting dependence (1), into conservation of momentum, continuity and energy equations, and neglecting the terms of body forces, thus after layer boundary simplifications forces we obtain the following non-linear system of basic equations in the curvilinear orthogonal co-ordinates (α_1, α_2, α_3,) [4-7, 12, 13]:

$$0=-\frac{1}{h_1}\frac{\partial p}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\left[\eta_0\,\eta_1(\alpha_1,\alpha_3)\,\eta_\pi(\alpha_1,\alpha_2,\alpha_3)\frac{\partial v_1}{\partial\alpha_2}\right], \tag{2}$$

$$0=\frac{\partial p}{\partial\alpha_2}, \tag{3}$$

$$0=-\frac{1}{h_3}\frac{\partial p}{\partial\alpha_3}+\frac{\partial}{\partial\alpha_2}\left[\eta_0\,\eta_1(\alpha_1,\alpha_3)\,\eta_\pi(\alpha_1,\alpha_2,\alpha_3)\frac{\partial v_3}{\partial\alpha_2}\right], \tag{4}$$

$$0=\frac{1}{h_1}\frac{\partial v_1}{\partial\alpha_1}+\frac{\partial v_2}{\partial\alpha_2}+\frac{1}{h_1h_3}\frac{\partial}{\partial\alpha_3}(h_1v_3), \tag{5}$$

$$\frac{\partial}{\partial\alpha_2}\left(\kappa\frac{\partial T}{\partial\alpha_2}\right)+\eta_0\,\eta_1(\alpha_1,\alpha_3)\,\eta_\pi(\alpha_1,\alpha_2,\alpha_3)\left[\left(\frac{\partial v_1}{\partial\alpha_2}\right)^2+\left(\frac{\partial v_3}{\partial\alpha_2}\right)^2\right]=0, \tag{6}$$

where the length, width and gap- height directions, are limited respectively: $0<\alpha_1\le 2\pi$, $-b_m\le$ $\le\alpha_3\le b_s$, $0\le\alpha_2\le\varepsilon$.

The system of Eq. (2)-(6) contains the following dimensional unknowns: pressure $p(\alpha_1,\alpha_3)$, temperature $T(\alpha_1,\alpha_2,\alpha_3)$, three oil velocity components $v_i(\alpha_1,\alpha_2,\alpha_3)$, for $i=1,2,3$ in three curvilinear, orthogonal dimensional directions: α_1, α_2, α_3. Lame coefficients are as follows: $h_1(\alpha_3)$, $h_3(\alpha_3)$ for rotational surfaces and non-monotone curvatures. The lubricant flow in bearing gap is generated by rotation of a rotational, curvilinear journal surface. Total gap height ε_T changes in direction α_2. Boundary conditions for lubricant velocity components have the form:

$$v_1=\omega h_1 \text{ for } \alpha_2=0,\; v_1=0 \text{ for } \alpha_2=\varepsilon_\mathrm{T}, \tag{7}$$

$$v_2=0 \text{ for } \alpha_2=0,\; v_2=0 \text{ for } \alpha_2=\varepsilon_\mathrm{T}, \tag{8}$$

$$v_3=0 \text{ for } \alpha_2=0,\; v_2=0 \text{ for } \alpha_2=\varepsilon_\mathrm{T}. \tag{9}$$

3. Integration Methods

Solutions of the partial differential equations (2)-(4) under the boundary conditions (7)-(8), have the following form [12]:

$$v_1(\alpha_1,\alpha_2,\alpha_3)=\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}A_\eta+(1-A_s)\,\omega\, h_1, \tag{10}$$

$$v_3(\alpha_1,\alpha_2,\alpha_3)=\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3}A_\eta, \tag{11}$$

where:

$$A_s(\alpha_1,\alpha_2,\alpha_3)\equiv\frac{\int\limits_0^{\alpha_2}\frac{1}{\eta_\Sigma}\mathrm{d}\alpha_2}{\int\limits_0^{\varepsilon_T}\frac{1}{\eta_\Sigma}\mathrm{d}\alpha_2},\quad A_\eta(\alpha_1,\alpha_2,\alpha_3)\equiv\int\limits_0^{\alpha_2}\frac{\alpha_2}{\eta_\Sigma}\mathrm{d}\alpha_2-A_s(\alpha_1,\alpha_2,\alpha_3)\int\limits_0^{\varepsilon_T}\frac{\alpha_2}{\eta_\Sigma}\mathrm{d}\alpha_2, \tag{12}$$

where: $0\le\alpha_1\le 2\pi\theta_1$, $0\le\theta_1\le 1$, $b_m\le\alpha_3\le b_s$, $0\le\alpha_2\le\varepsilon_T$, $\varepsilon_T=\varepsilon_T(\alpha_1,\alpha_3)$, $\eta_\Sigma(\alpha_1,\alpha_2,\alpha_3)$.

Solutions of the continuity equations (5) under the boundary conditions (8) where $v_2=0$ for $\alpha_2=0$, leads to the following form:

$$v_2(\alpha_1,\alpha_2,\alpha_3)=-\int\limits_0^{\alpha_2}\frac{1}{h_1}\frac{\partial v_1}{\partial \alpha_1}\mathrm{d}\alpha_2-\int\limits_0^{\alpha_2}\frac{1}{h_1 h_3}\frac{\partial (h_1 v_3)}{\partial \alpha_3}\mathrm{d}\alpha_2. \tag{13}$$

where: $0\le\alpha_1\le 2\pi\theta_1$, $0\le\theta_1\le 1$, $b_m\le\alpha_3\le b_s$, $0\le\alpha_2\le\varepsilon_T$.

Putting the formula (1) into formulae (10)-(12) we obtain:

$$A_s(\alpha_1,\alpha_2,\alpha_3)\equiv\frac{1}{\Omega_1}\int\limits_0^{\alpha_2}\eta_\pi\,\mathrm{d}\alpha_2,\quad A_\eta(\alpha_1,\alpha_2,\alpha_3)\equiv\frac{1}{\eta_0\,\eta_1}\left[\int\limits_0^{\alpha_2}\alpha_2\,\eta_\pi\,\mathrm{d}\alpha_2-\frac{\Omega_2}{\Omega_1}\int\limits_0^{\alpha_2}\eta_\pi\,\mathrm{d}\alpha_2\right],$$

$$\frac{\partial A_s(\alpha_1,\alpha_2,\alpha_3)}{\partial \alpha_2}\equiv\frac{1}{\Omega_1}\eta_\pi,\quad \frac{\partial A_\eta(\alpha_1,\alpha_2,\alpha_3)}{\partial \alpha_2}\equiv\frac{1}{\eta_0\,\eta_1}\eta_\pi\left(\alpha_2-\frac{\Omega_2}{\Omega_1}\right), \tag{14}$$

where:

$$\Omega_1(\alpha_1,\alpha_3)\equiv\int\limits_0^{\varepsilon_T}\eta_\pi\,\mathrm{d}\alpha_2,\quad \Omega_2(\alpha_1,\alpha_3)\equiv\int\limits_0^{\varepsilon_T}\alpha_2\,\eta_\pi\,\mathrm{d}\alpha_2. \tag{15}$$

Putting the results (14) into the first derivatives in α_2 direction of velocity components (12), (11), we obtain:

$$\frac{\partial v_1(\alpha_1,\alpha_2,\alpha_3)}{\partial \alpha_2}=\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}\frac{\partial A_\eta}{\partial \alpha_2}-\omega\, h_1\frac{\partial A_s}{\partial \alpha_2}=\eta_\pi\left[\frac{1}{\eta_0\,\eta_1\,h_1}\frac{\partial p}{\partial \alpha_1}\left(\alpha_2-\frac{\Omega_2}{\Omega_1}\right)-\frac{\omega\, h_1}{\Omega_1}\right], \tag{16}$$

$$\frac{\partial v_3(\alpha_1,\alpha_2,\alpha_3)}{\partial \alpha_2}=\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3}\frac{\partial A_\eta}{\partial \alpha_2}=\frac{\eta_\pi}{\eta_0\,\eta_1\,h_3}\frac{\partial p}{\partial \alpha_3}\left(\alpha_2-\frac{\Omega_2}{\Omega_1}\right). \tag{17}$$

Substituting expressions (16), (17) into the energy equation (6), after terms ordering and calculations, the energy equation tends to the following form:

$$\frac{\partial^2 T}{\partial \alpha_2^2}+\eta_\pi(F\alpha_2^2+G\,\alpha_2+K)=0, \tag{18}$$

where:

$$F \equiv \frac{1}{\eta_0\,\eta_1\,\kappa}\Pi_p,\;\; G \equiv \frac{-2}{\eta_0\,\eta_1\,\kappa}\left(\Pi_p + \frac{\omega h_1\,\eta_0\,\eta_1}{\Omega_2}\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}\right)\frac{\Omega_2}{\Omega_1}, \tag{19}$$

$$K \equiv \frac{1}{\eta_0\,\eta_1\,\kappa}\left[\Pi_p + \frac{2\omega h_1\,\eta_0\,\eta_1}{\Omega_2}\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1} + \left(\frac{\omega h_1\,\eta_0\,\eta_1}{\Omega_2}\right)^2\right]\left(\frac{\Omega_2}{\Omega_1}\right)^2, \tag{20}$$

$$\Pi_p \equiv \left(\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1}\right)^2 + \left(\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3}\right)^2. \tag{21}$$

Now we put the solutions (10), (11) in the solution (13). If we impose second boundary condition (8) for radial component of lubricant velocity i.e. $v_2 = 0$ for $\alpha_2 = \varepsilon_T$ then it is easy to find out that the pressure function p in the curvilinear coordinates (α_1, α_2, α_3) satisfies the following modified Reynolds equation:

$$\frac{1}{h_1}\frac{\partial}{\partial \alpha_1}\left[\frac{\partial p}{\partial \alpha_1}\left(\int_0^{\varepsilon_T} A_\eta\,\mathrm{d}\alpha_2\right)\right] + \frac{1}{h_3}\frac{\partial}{\partial \alpha_3}\left[\frac{h_1}{h_3}\frac{\partial p}{\partial \alpha_3}\left(\int_0^{\varepsilon_T} A_\eta\,\mathrm{d}\alpha_2\right)\right] = \omega h_1\frac{\partial}{\partial \alpha_1}\left[\int_0^{\varepsilon_T} A_s\,\mathrm{d}\alpha_2 - \varepsilon_T\right], \tag{22}$$

where oil dynamic viscosity has varies values in gap height direction, i.e. $\eta(\alpha_1, \alpha_2, \alpha_3)$.

On the ground of the assumption (1), we ought integrate the expressions (14) to obtain:

$$\int_0^{\varepsilon_T} A_\eta\,\mathrm{d}\alpha_2 \equiv \frac{1}{\eta_0\,\eta_1}\left[\int_0^{\varepsilon_T}\int_0^{\alpha_2} \alpha_2\,\eta_\pi\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2 - \frac{\Omega_2}{\Omega_1}\int_0^{\varepsilon_T}\int_0^{\alpha_2} \eta_\pi\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2\right] = \frac{1}{\eta_0\,\eta_1}\int_0^{\varepsilon_T}\int_0^{\alpha_2} \eta_\pi\left(\alpha_2 - \frac{\Omega_2}{\Omega_1}\right)\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2, \tag{23}$$

$$\int_0^{\varepsilon_T} A_s\,\mathrm{d}\alpha_2 \equiv \frac{1}{\Omega_1}\int_0^{\varepsilon_T}\int_0^{\alpha_2} \eta_\pi\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2. \tag{24}$$

Putting integrals (23), (24) into (22), we obtain finally modified Reynolds equation. Such form of pressure equation with the mutually coupled temperature equation (18) describe the set of two equations presenting the equivalent form of non-isothermal solution in the following form:

$$\begin{aligned}
&\frac{1}{h_1}\frac{\partial}{\partial \alpha_1}\left\{\frac{\partial p}{\partial \alpha_1}\frac{1}{\eta_0\,\eta_1}\left[\int_0^{\varepsilon_T}\int_0^{\alpha_2} \eta_\pi\left(\alpha_2 - \frac{\Omega_2}{\Omega_1}\right)\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2\right]\right\} + \\
&+\frac{1}{h_3}\frac{\partial}{\partial \alpha_3}\left\{\frac{h_1}{h_3}\frac{\partial p}{\partial \alpha_3}\frac{1}{\eta_0\,\eta_1}\left[\int_0^{\varepsilon_T}\int_0^{\alpha_2} \eta_\pi\left(\alpha_2 - \frac{\Omega_2}{\Omega_1}\right)\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2\right]\right\} = \\
&= \omega h_1\frac{\partial}{\partial \alpha_1}\left[\frac{1}{\Omega_1}\int_0^{\varepsilon_T}\int_0^{\alpha_2} \eta_\pi\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_2 - \varepsilon_T\right],
\end{aligned} \tag{25a}$$

$$\frac{\partial^2 T}{\partial \alpha_2^2} + \eta_\pi[F(p,T)\alpha_2^2 + G(p,T)\alpha_2 + K(p,T)] = 0, \tag{25b}$$

where: $0 \le \alpha_1 \le 2\pi\theta_1$, $0 \le \theta_1 \le 1$, $b_m \le \alpha_3 \le b_s$, $0 \le \alpha_2 \le \varepsilon_T$.

The non-linear system of two mutually coupled partial differential equations (25a) + (25b) determines unknown functions of pressure p and temperature T.

Remark: The solution $p(T$, pH,) of modified Reynolds equation (25a) depends of the solution T of modified Energy equation (25b), and additionally of the pH as well adhesion as capillary forces.

Simultaneously the reciprocal relation, namely the solutions of modified Energy equation i.e. temperature $T(p$, pH,…) depends on the solution of modified Reynolds equation i.e. on the pressure p and additionally of the pH, adhesion, capillary forces.

4. Particular case for constant oil dynamic viscosity independent on temperature

For constant dynamic viscosity in gap height direction i.e. for $\eta_\pi = 1$, from (14), (15), (23), (24) we obtain:

$$A_s = \frac{\alpha_2}{\varepsilon_T} \equiv s, \quad \int_0^{\varepsilon_\mathrm{T}} A_\eta \,\mathrm{d}\alpha_2 = -\frac{\varepsilon_\mathrm{T}^3}{12\eta_0\,\eta_1}, \quad \int_0^{\varepsilon_\mathrm{T}} A_s \,\mathrm{d}\alpha_2 - \varepsilon_\mathrm{T} = -\frac{1}{2}\varepsilon_\mathrm{T}, \quad \Omega_1 = \varepsilon_\mathrm{T}, \quad \Omega_2 = \frac{\varepsilon_\mathrm{T}^2}{2}. \tag{26}$$

Hence the modified Reynolds equation (25) tends to the following almost classical dimensional form in curvilinear orthogonal coordinates:

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left[\frac{\varepsilon_\mathrm{T}^3}{\eta_0\,\eta_1}\frac{\partial p}{\partial\alpha_1}\right] + \frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1}{h_3}\frac{\varepsilon_\mathrm{T}^3}{\eta_0\,\eta_1}\frac{\partial p}{\partial\alpha_3}\right] = 6\omega\, h_1\frac{\partial\varepsilon_\mathrm{T}}{\partial\alpha_1}. \tag{27}$$

For constant dynamic viscosity in gap height direction i.e. for $\eta_\pi = 1$, the equations (18), (19), (20) describing the temperature distribution, tend to the following classical dimensional forms:

$$\frac{\partial^2 T}{\partial\alpha_2^2} + F(\eta_\pi = 1)\,\alpha_2^2 + G(\eta_\pi = 1)\alpha_2 + K(\eta_\pi = 1) = 0, \tag{28}$$

with:

$$F(\eta_\pi = 1) \equiv \frac{1}{\eta_0\,\eta_1\,\kappa}\Pi_p, \quad G(\eta_\pi = 1) \equiv \frac{-\varepsilon_\mathrm{T}}{\eta_0\,\eta_1\,\kappa}\left(\Pi_p + \frac{2\omega\, h_1\,\eta_0\,\eta_1}{\varepsilon_\mathrm{T}^2}\frac{1}{h_1}\frac{\partial p}{\partial\alpha_1}\right), \tag{29}$$

$$K(\eta_\pi = 1) \equiv \frac{\varepsilon_\mathrm{T}^2}{\eta_0\,\eta_1\,\kappa}\left[\frac{1}{4}\Pi_p + \left(\frac{\omega\, h_1\,\eta_0\,\eta_1}{\varepsilon_\mathrm{T}^2}\right)\frac{1}{h_1}\frac{\partial p}{\partial\alpha_1} + \left(\frac{\omega\, h_1\,\eta_0\,\eta_1}{\varepsilon_\mathrm{T}^2}\right)^2\right]. \tag{30}$$

Now the resolving system of non-isothermal lubrication problem we can show in the form of the set of two semi-independent equations namely modified Reynolds and modified Energy equations in following form:

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left[\frac{\varepsilon_\mathrm{T}^3}{\eta_0\,\eta_1}\frac{\partial p}{\partial\alpha_1}\right] + \frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1}{h_3}\frac{\varepsilon_\mathrm{T}^3}{\eta_0\,\eta_1}\frac{\partial p}{\partial\alpha_3}\right] = 6\omega\, h_1\frac{\partial\varepsilon_\mathrm{T}}{\partial\alpha_1}, \tag{31a}$$

$$\frac{\partial^2 T}{\partial\alpha_2^2} + F(\eta_\pi = 1, p)\,\alpha_2^2 + G(\eta_\pi = 1, p)\,\alpha_2 + K(\eta_\pi = 1, p) = 0. \tag{31b}$$

The non-linear system of two semi-independent partial differential equations (31a) + (31b) determines unknown functions of pressure p and temperature T.

The pressure solution p of modified Reynolds equation (31a) is independent of the temperature solutions T of modified Energy equation (31b), but reciprocal relation non-valid, because the solutions of modified Energy equation i.e. temperature $T(p)$ depends on the solution of modified Reynolds equation i.e. on pressure function p.

In despite that the influences of the temperature, hydrogen ion concentrations and influences of adhesion and capillarity forces on the lubricant dynamic viscosity variations in gap height direction are neglected i.e. dynamic viscosity is constant across the film thickness, yet the temperature of the oil varies in gap height direction. Hence, in this case the energy equation (31b) and pressure equation (31a) are not mutually coupled.

5. Conclusions

The achievements obtained in this paper for the real exact analytical 3D solutions for non-isothermal, stationary, laminar hydrodynamic lubrication problem with 3D changes of lubricant dynamic viscosity in curvilinear orthogonal coordinates, are as follows:

- Primary assumption of constant lubricant dynamic viscosity across the film thickness leads to the quasi conjugation of the pressure and energy equations whereas we have:
  - Pressure independent on temperature and temperature depended on pressure.
  - The constant viscosity across the film thickness not exclude the temperature variations in gap height direction what denotes the contradiction respect to the primary assumptions.
- Primary assumption of constant temperature across the film thickness is contradictory to the layer boundary simplifications of isothermal hydrodynamic equation describing the lubrication because the gradients of temperature in gap height direction are comparable with the gradients of temperature in two remaining directions.
- Primary assumption of variable lubricant dynamic viscosity across the film thickness leads to the full conjugation of the pressure and energy equations whereas we have:
  - Pressure dependent on temperature and temperature depended on pressure.
  - The variable viscosity across the film thickness leads to the temperature variations in gap height direction what confirm the primary assumptions.
- Solutions presented in curvilinear orthogonal coordinates enable the application of obtained results for hydrodynamic bearing and biobearing with various shapes of rotating journals and motionless sleeves.

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