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AN APPLICATION OF STIFFNESS MATRIX METHOD TO DETERMINING OF DISPERSION CURVES FOR ARBITRARY COMPOSITE MATERIALS

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Abstract

Nowadays multi-layered composite material is very often applied in different kind of structures, like aircrafts, boats or vehicles. Parts of structures, which are made of these materials, are significantly lighter in comparison with traditional materials, like aluminum or steel alloys. On the other hand, the process of damage creation and evolution in the case of composites is much more complex. Moreover, the damages, which are characteristic for multi-layered materials (matrix cracking, fibre breakage, delaminations), are very difficult to detect at early stage of creation. Hence, there is a need to develop the advanced methods to detect them without destroying tested composite element. One of them is based on analysis of elastic wave propagation through the composite structure. Unfortunately, elastic waves possess strongly dispersive character. Thus, it is necessary to determine dispersion curves for investigated material before the tests in order to appropriate interpretation of received dynamic response of structure. In the case of arbitrary composite materials, it is rather challenging task. In the present article the relatively new, analytical method is applied, namely stiffness matrix method. The fundamental assumptions and the theoretical formulation of this method are discussed. Next numerical examples are presented, namely the dispersion curves are determined for the single orthotropic lamina and multi-layered 'quasi - isotropic' composite plate. The studied plates are made of glass fibres and epoxy resin. In the case of single lamina, the dispersion curves are determined in the parallel, perpendicular and arbitrary direction of waves propagation with respect to the fibre direction. In the case of multi-layered plates, the dispersion curves are computed for one arbitrary direction. Additionally, the phase and group velocities for fundamental modes and fixed excitation frequency are estimated in all directions of waves propagation.

Keywords: *SHM system, multi-layered composites, elastic waves, dispersion curves, stiffness matrix method*

1. Introduction

Presently, the majority of engineering structures are made of traditional isotropic materials, like, for example, steel or aluminium alloys. However, these materials are very often replaced by multi-layered composite materials. In the case of composites, the process of damage formation is very complex (matrix cracking, fibre breakage, delaminations). It should be stressed here that most of flaws are not visible in direct observation, especially at the early stage of formation. Thus, they have to be detected with the use of advanced method without destruction of studied structures or elements. Generally, these methods are known in literature as non-destructive testing. Among different possibilities, it seems that the use of guided waves propagation is very promising, mainly due to the fact that these waves can travel through the structures for long distances [1]. However, the elastic waves are strongly dispersive and they have a multimodal character. It causes that the damage detection based on an analysis of dynamic response of interrogated structure is rather difficult. Therefore, determination of dispersion curves is one of the most important aspect of all systems for damage detecting especially in the case of composite materials. Generally, in the case of multi-layered materials determination of the dispersion curves is rather a difficult task. There are available three different method. The first of them, known as the transfer matrix method, was proposed by Thompson [13] in 1950 and next corrected by Haskell [2] in 1953. Initially this approach was adopted for the composites where all layers have isotropic mechanical properties.

Next, Nayfeh [9, 3] extended this approach to the case of arbitrary composite materials. This approach is relatively simple and easy to use. However, the transfer matrix method is numerically unstable for relatively large product of frequency and thickness of composite wall. It is well known as fd problem, Lowe [7]. An alternative to this is the global matrix method approach proposed by Knopoff [6] in 1964. This method is used in the case of anisotropic composite materials by Pant et al. [10] .Unfortunately; it seems that in this method the problem of numerical instabilities is also present Lowe [7]. Moreover, in this method, the dynamic properties of a composite are described by a single matrix, which size strictly depends on number of layers. Thus in the case of composites, where the number of layers is large, the computations could be very time consuming. In order to avoid numerical instabilities, which are main disadvantage of the mentioned above methods, Kausel [5] in 1986 and further Wang and Rokhlin [14, 11, 12] in 2001 proposed the stiffness matrix method. The main idea of this relatively new approach is to remove the exponential terms from the diagonal of the matrix, which describes the dynamical properties of structure. The numerically unstable transfer matrix is replaced by the stiffness matrix, which relates the components of stress at the bottom and top of the layer with the displacement at the bottom and top layer. The stiffness matrix for the whole composite is obtained with the use of advanced recursive algorithm. It should be stressed here that this method is unconditionally stable and only slight less efficient in comparison with transfer matrix method. This method is also used by Kamal and Giurgiutiu [4] in the case of arbitrary multi-layered composites.

In this present article, the stiffness matrix method is used to obtain the dispersion curves for the composite, which is made of glass fibres GFRP E-glass and epoxy resin [8]. The calculations are carried out for a single orthotropic lamina as well as a multi-layered composite material with the following ply orientations [0°, 90°, 45°, –45°, –45°, 45°, 90°, and 0°].

2. Stiffness Matrix Method

Generally, it is assumed that the elastic waves propagate in the direction, which is parallel to the x_1-x_3 plane of the global Cartesian coordinate system. The wave solution describing displacement components u_i can be written as follows [1]:

$$(u_1, u_2, u_3) = (U_1, U_2, U_3)e^{i\xi(x_1 + \alpha x_3 - ct)}. \tag{1}$$

In the above expressions U_i are the unknown displacement amplitudes of partial waves, $i^2=-1$ is the imaginary unit, $\xi=\omega/c$ denotes the wave number, c is the phase velocity, ω is the circular frequency $\omega=2\pi f$ and t is time. Here it should be stressed that according to Snell's law, all partial waves have the same frequency f. The parameter α will be determined later. For the k-th layer the expression, which relates the stress at the bottom and top layer with the displacement at the top and bottom layer, can be written in the following form, namely:

$$\begin{Bmatrix} \{\sigma\}_{k-1} \\ \{\sigma\}_k \end{Bmatrix} = [A]_k [B]_k^{-1} \begin{Bmatrix} \{u\}_{k-1} \\ \{u\}_k \end{Bmatrix} = [K]_k \begin{Bmatrix} \{u\}_{k-1} \\ \{u\}_k \end{Bmatrix}, \tag{2}$$

where subscript 'k-1' means the top surface of the k-th layer and subscript k means the bottom surface of the k-th layer. Further, {σ}, {u} denotes the stress and displacement, respectively. The matrixes $[A]_k$ and $[B]_k$ takes the form [4]:

$$[A]_k = \begin{bmatrix} D_{11} & D_{13} & D_{15} & D_{11}e^{i\xi\alpha_1 d_k} & D_{13}e^{i\xi\alpha_3 d_k} & D_{15}e^{i\xi\alpha_5 d_k} \\ D_{21} & D_{23} & D_{25} & -D_{21}e^{i\xi\alpha_1 d_k} & -D_{23}e^{i\xi\alpha_2 d_k} & -D_{25}e^{i\xi\alpha_5 d_k} \\ D_{31} & D_{33} & D_{35} & -D_{31}e^{i\xi\alpha_1 d_k} & -D_{33}e^{i\xi\alpha_2 d_k} & -D_{35}e^{i\xi\alpha_5 d_k} \\ D_{11}e^{i\xi\alpha_1 d_k} & D_{13}e^{i\xi\alpha_3 d_k} & D_{15}e^{i\xi\alpha_5 d_k} & D_{11} & D_{13} & D_{15} \\ D_{21}e^{i\xi\alpha_1 d_k} & D_{23}e^{i\xi\alpha_3 d_k} & D_{25}e^{i\xi\alpha_5 d_k} & D_{21} & D_{23} & D_{25} \\ D_{31}e^{i\xi\alpha_1 d_k} & D_{33}e^{i\xi\alpha_3 d_k} & D_{35}e^{i\xi\alpha_5 d_k} & D_{31} & D_{33} & D_{35} \end{bmatrix}, \tag{3}$$

$$[B]_k = \begin{bmatrix} 1 & 1 & 1 & e^{i\xi\alpha_1 d_k} & e^{i\xi\alpha_3 d_k} & e^{i\xi\alpha_5 d_k} \\ V_1 & V_3 & V_5 & V_1 e^{i\xi\alpha_1 d_k} & V_3 e^{i\xi\alpha_2 d_k} & V_5 e^{i\xi\alpha_5 d_k} \\ W_1 & W_3 & W_5 & -W_1 e^{i\xi\alpha_1 d_k} & -W_3 e^{i\xi\alpha_2 d_k} & -W_5 e^{i\xi\alpha_5 d_k} \\ e^{i\xi\alpha_1 d_k} & e^{i\xi\alpha_3 d_k} & e^{i\xi\alpha_5 d_k} & 1 & 1 & 1 \\ V_1 e^{i\xi\alpha_1 d_k} & V_3 e^{i\xi\alpha_3 d_k} & V_5 e^{i\xi\alpha_5 d_k} & V_1 & V_3 & V_5 \\ W_1 e^{i\xi\alpha_1 d_k} & W_3 e^{i\xi\alpha_3 d_k} & W_5 e^{i\xi\alpha_5 d_k} & -W_1 & -W_3 & -W_5 \end{bmatrix}. \tag{4}$$

where d_k denotes the thickness of the layer. The elements of matrix $[A]_k$ and $[B]_k$ are described by the following relationships, namely:

$$D_{1j} = C_{13} + C_{36}V_j + C_{33}W_j\alpha_j,\ D_{2j} = C_{55}(\alpha_j + W_j) + C_{45}V_j\alpha_j,\ D_{3j} = C_{45}(\alpha_j + W_j) + C_{44}V_j\alpha_j, \tag{5}$$

$$V_j = \frac{U_{2j}}{U_{1j}} = \frac{K_{13}(\alpha_j)K_{23}(\alpha_j) - K_{12}(\alpha_j)K_{33}(\alpha_j)}{K_{22}(\alpha_j)K_{33}(\alpha_j) - K_{23}(\alpha_j)K_{23}(\alpha_j)},\ W_j = \frac{U_{3j}}{U_{1j}} = \frac{K_{12}(\alpha_j)K_{23}(\alpha_j) - K_{13}(\alpha_j)K_{22}(\alpha_j)}{K_{22}(\alpha_j)K_{33}(\alpha_j) - K_{23}(\alpha_j)K_{23}(\alpha_j)}, \tag{6}$$

where C_{ij} are the components of the stiffness matrix, which describes the mechanical properties of the layer after transformation from the layer to the global coordinate system. The amplitudes of partial waves U_{ij} can be determined from the following system of linear equations:

$$[K(\alpha)]\{U\} = 0, \tag{7}$$

where:

$$[K(\alpha)] = \begin{bmatrix} C_{11} - \rho c^2 + C_{55}\alpha^2 & C_{16} + C_{45}\alpha^2 & (C_{13} + C_{55})\alpha \\ C_{16} + C_{45}\alpha^2 & C_{66} - \rho c^2 + C_{44}\alpha^2 & (C_{36} + C_{45})\alpha \\ (C_{13} + C_{55})\alpha & (C_{36} + C_{45})\alpha & C_{55} - \rho c^2 + C_{33}\alpha^2 \end{bmatrix}. \tag{8}$$

The symbol ρ in (8) denotes the density of the layer material. In order to obtain non-trivial solution of (7), the determinant of the coefficient matrix (8) has to be equal to zero. To fulfil this condition the following 6-th order polynomial equation with respect to the scalar parameter α is obtained, namely:

$$A\alpha^6 + B\alpha^4 + C\alpha^2 + D = 0. \tag{9}$$

There are six real or complex roots of this equation, namely $\alpha_1 = -\alpha_2$, $\alpha_3 = -\alpha_4$ and $\alpha_5 = -\alpha_6$. In order to obtain the stiffness matrix for the completely composite material, an advanced recursive algorithm has to be applied [11]. Let us consider two adjoining layers (1, 2), namely:

$$\begin{Bmatrix} \{\sigma\}_0 \\ \{\sigma\}_1 \end{Bmatrix} = \begin{bmatrix} [K]^A_{11} & [K]^A_{12} \\ [K]^A_{21} & [K]^A_{22} \end{bmatrix} \begin{Bmatrix} \{u\}_0 \\ \{u\}_1 \end{Bmatrix},\quad \begin{Bmatrix} \{\sigma\}_1 \\ \{\sigma\}_2 \end{Bmatrix} = \begin{bmatrix} [K]^B_{11} & [K]^B_{12} \\ [K]^B_{21} & [K]^B_{22} \end{bmatrix} \begin{Bmatrix} \{u\}_1 \\ \{u\}_2 \end{Bmatrix}, \tag{10}$$

where subscripts denote the interfaces. By excluding $\{\sigma\}_1$ and $\{u\}_1$ from the first relation and substituting in the second one, the matrix, which relates $\{\sigma\}_0$ $\{u\}_0$ to $\{\sigma\}_2$ $\{u\}_2$, is obtained. This combined matrix is a stiffness matrix for these two bonded layers:

$$\begin{Bmatrix} \{\sigma\}_0 \\ \{\sigma\}_2 \end{Bmatrix} = \begin{bmatrix} [K]^A_{11} + [K]^A_{12}\left([K]^B_{11} - [K]^A_{22}\right)^{-1}[K]^A_{21} & -[K]^A_{12}\left([K]^B_{11} - [K]^A_{22}\right)^{-1}[K]^B_{12} \\ [K]^B_{21}\left([K]^B_{11} - [K]^A_{22}\right)^{-1}[K]^A_{21} & [K]^B_{22} - [K]^B_{21}\left([K]^B_{11} - [K]^A_{22}\right)^{-1}[K]^B_{12} \end{bmatrix} \begin{Bmatrix} \{u\}_0 \\ \{u\}_2 \end{Bmatrix}. \tag{11}$$

Denoting the stiffness matrix obtained by $[K]^A$ and the stiffness matrix for the third layer by $[K]^B$, we can recursively apply the relation (11) to obtain the global stiffness matrix, which relates the stresses to the displacement for the top and bottom surface of the whole composite plate. The wave characteristic equation for the completely composite structure is obtained from the total stiffness matrix. Assuming that the components of stress on the top and bottom surface are equal to zero, the Lamb wave dispersion equation is:

$$\det([K]) = 0. \tag{12}$$

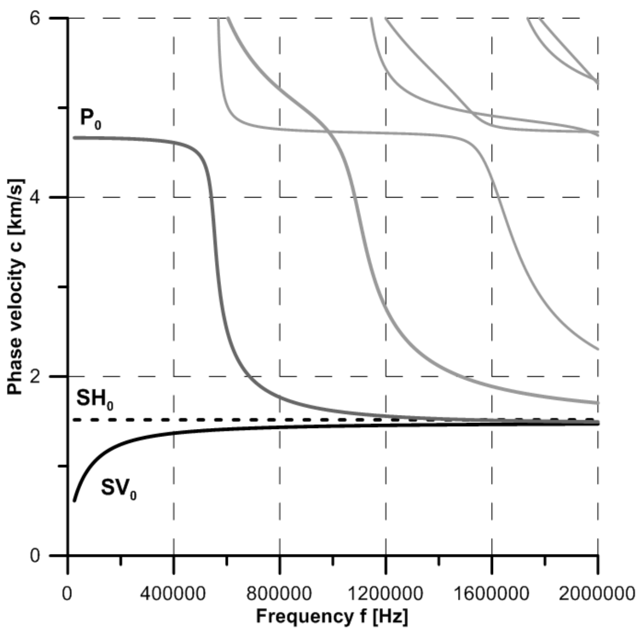
In other words, for the assumed value of frequency ω the value of phase velocity c is looked for. According to the authors' experience, it seems that in order to find the roots of (12), the bisection method is the most suitable.

3. Dispersion curves

The dispersion curves are determined for the single lamina as well as for the composite, which consists of 8 layers with following ply orientation [0°, 90°, 45°, –45°, –45°, 45°, 90°, 0°]. The total thickness of plates are t_c=2 mm in both cases. In the case of multi-layered plate all layers have identical thickness t_l=0.25 mm. The studied structures are made of identical material, namely glass fibres GFRP E-glass and epoxy resin [8]. The mechanical properties of layer material are as follows: E_1=38.6 GPa, E_2=8.27 GPa, G_{12}=7.17 GPa, ν_{12}=28 and density ρ=1.8 g/cm³. The dispersion curves are determined in the following frequency range 25 kHz ≤ f ≤ 2000 kHz and phase velocity range 0 ≤ c ≤ 6 km/s. All necessary numerical calculation is carried out with the use of SCILAB free software.

Single lamina

In the Fig. 1 there are depicted the dispersion curves, which are obtained for the single lamina. It is assumed that the elastic waves travel in the direction, which is parallel to the axis x_1 of the principal orthotropic direction (φ=0°). As it can be observed, the fundamental mode SV_0 in the case of low frequencies (f < 400 kHz) is strongly dispersive. However, for the higher frequencies its phase velocity is almost constant and is equal to about c≈1.47 km/s. The phase velocity of the shear horizontal mode SH_0 is constant in the studied range of frequencies. Its value is equal to c=152 km/s. The phase velocity of the symmetric mode P_0 for the low frequencies (f < 520 kHz) varies not significantly and its value c≈4.66 km/s. Next, its value suddenly decreases and finally is equal to c≈1.49 km/s. In the studied range of frequency, there are also 6 higher modes.



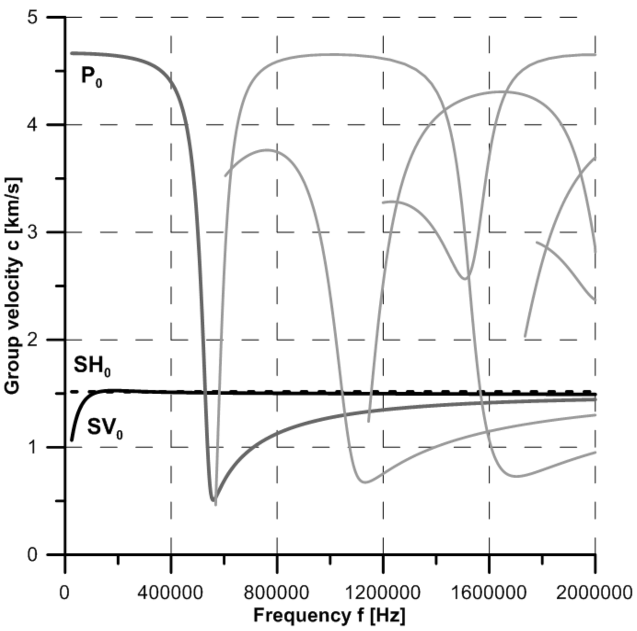


Fig. 1. Dispersion curves (phase and group velocities) for single glass/epoxy resin lamina. Total thickness of layer t_c=2 mm. Waves propagation angle φ =0°

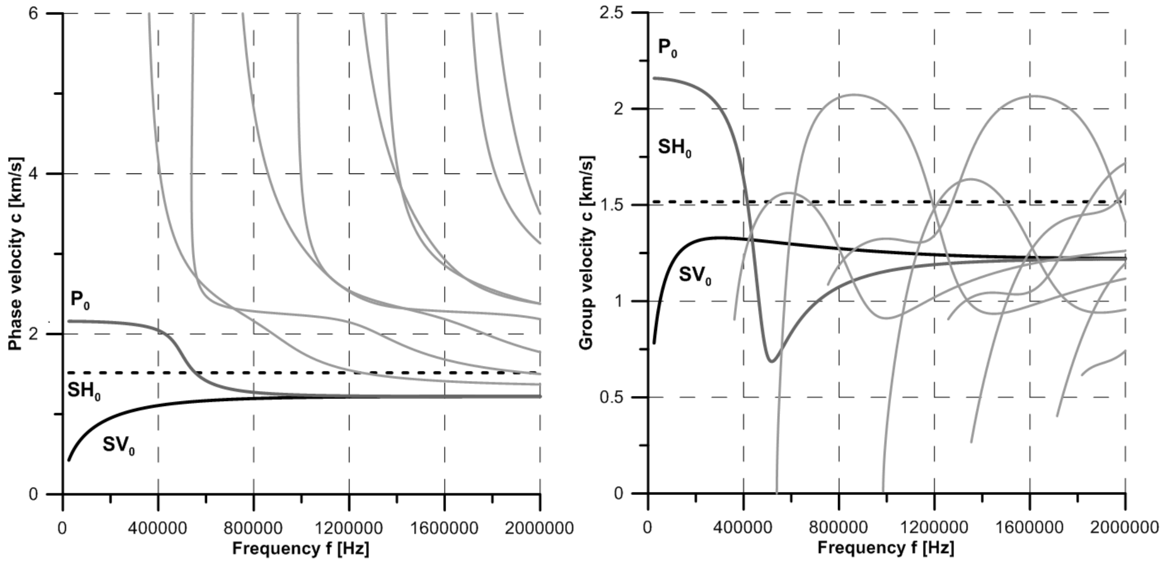


Fig. 2. Dispersion curves (phase and group velocities) for single glass/epoxy resin lamina. Total thickness of layer t_c=2 mm. Waves propagation angle φ =90°

The dispersion curves, which are obtained in the case, when the elastic waves propagate in the direction perpendicular to the fibres, are shown in the Fig. 2. Qualitative character of the presented curves is similar. However, the phase velocity of the P_0 and SV_0 for the higher frequencies is slightly reduced and now it is equal to c≈1.22 km/s. The initial phase velocity of the P_0 mode is significantly reduced to the value c≈2.16 km/s. It is worth noted that the phase velocity of the SH_0 mode is still constant (c=1.52 km/s). However, the number of higher modes increases and now the 8 modes are visible in the investigated range of frequency. However, in the case of the waves propagation angle (here φ=35°, Fig. 3) the obtained dispersion curves are quite different. The main difference is that the phase velocity of the SH_0 mode is not constant. Moreover, the number of higher modes is also different and now it is equal to 11. Moreover, in the Fig. 4 there are shown the phase and group velocities of the fundamental modes SV_0, SH_0 and P_0 with respect to the waves propagation angle. These graphs are prepared for the fixed frequency, where f = 250 kHz. It should be stressed that the phase as well as the group velocities of all fundamental modes strongly depends on the propagation direction, which is described by the angle φ.

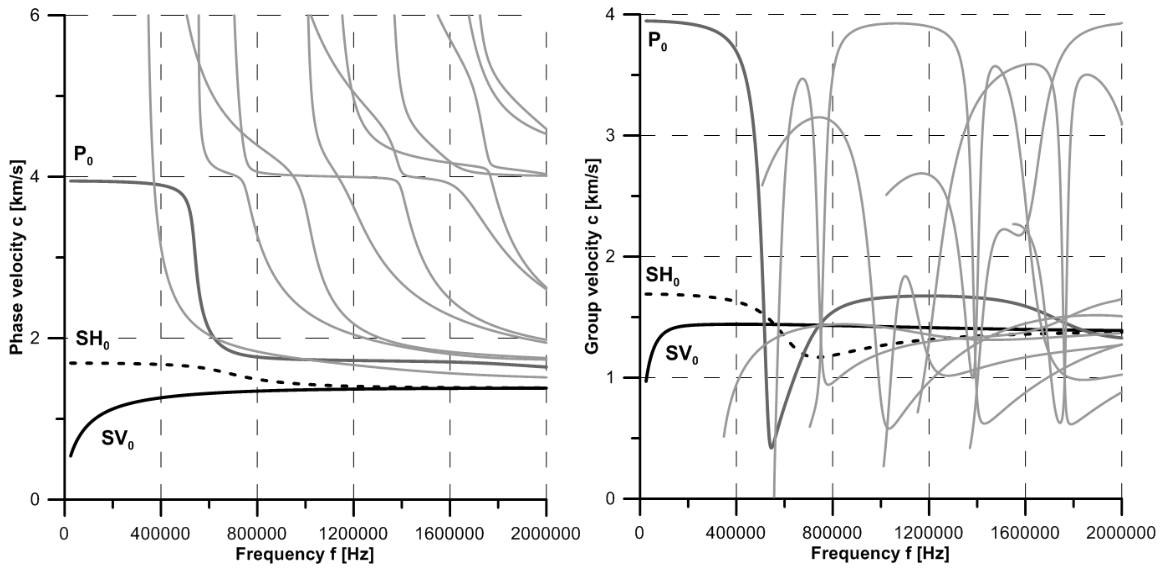


Fig. 3. Dispersion curves (phase and group velocities) for single glass/epoxy resin lamina. Total thickness of layer t_c=2 mm. Waves propagation angle φ =35°

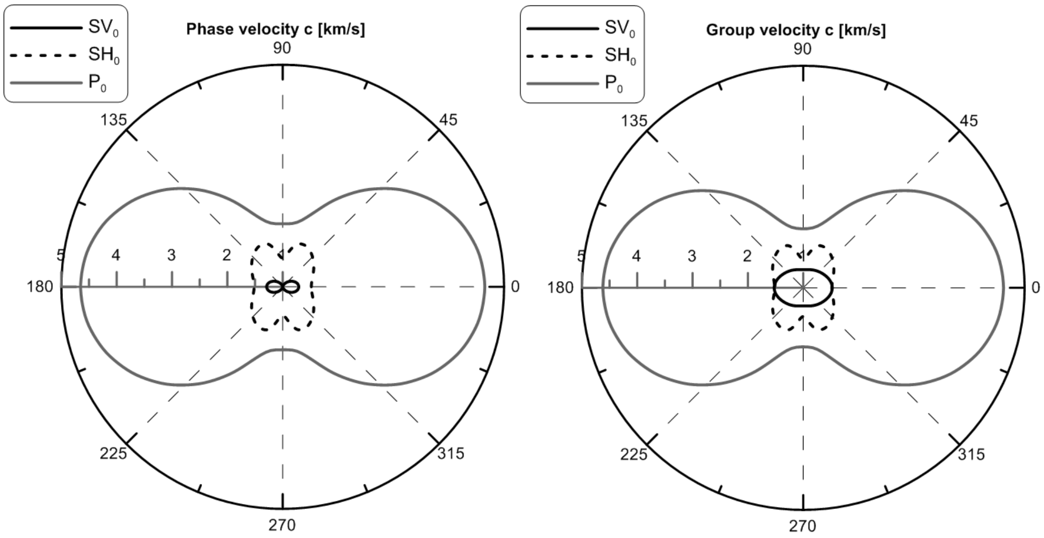


Fig. 4. Phase and group velocities with respect to waves angle propagation φ for single glass/epoxy resin lamina. Total thickness of layer t_c=2 mm. Fixed frequency f = 250 kHz

Composite [0°, 90°, 45°, –45°, –45°, 45°, 90°, 0°]

In the Fig. 5 there are depicted the dispersion curves, which are computed for the 'quasi-isotropic' multi-layered composite material. As before, the fundamental mode SV_0 for the higher frequencies is almost not dispersive. Its phase velocity is equal to about c≈1.39 km/s. The phase velocity of the shear horizontal SH_0 mode varies also significantly. The initial phase velocity of the SH_0 mode is equal to c=2.04 km/s. For the higher frequencies, this value is convergent to the phase velocity of the SV_0 mode. The fundamental mode P_0 behaves in the similar way as before. Initially its phase velocity is equal to c=3.37 km/s. However, in the case of higher frequencies the phase velocity is slightly greater (c≈1.56 km/s) in comparison with the phase velocities of the SV_0 and SH_0 modes. The relationship between the phase and group velocities of the fundamental modes and the waves propagation angle φ are presented in the Fig. 6. As before, these graphs are prepared for the frequency f=250 kHz. It should be stressed here that, as it is expected, the completely studied composite material has 'quasi-isotropic' mechanical properties. Thus, the phase and group velocities are almost insensitive of the angle φ. Moreover, the determined values of phase and group velocity are very similar.

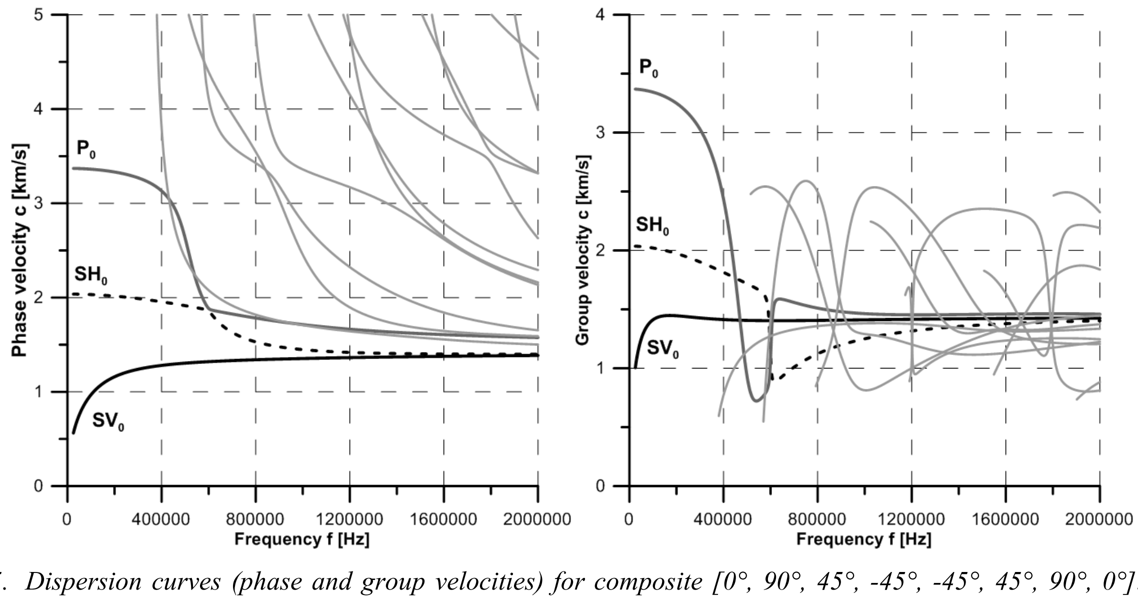


Fig. 5. Dispersion curves (phase and group velocities) for composite [0°, 90°, 45°, -45°, -45°, 45°, 90°, 0°]. Total thickness of composite t_c=2 mm. Waves propagation angle φ =0°

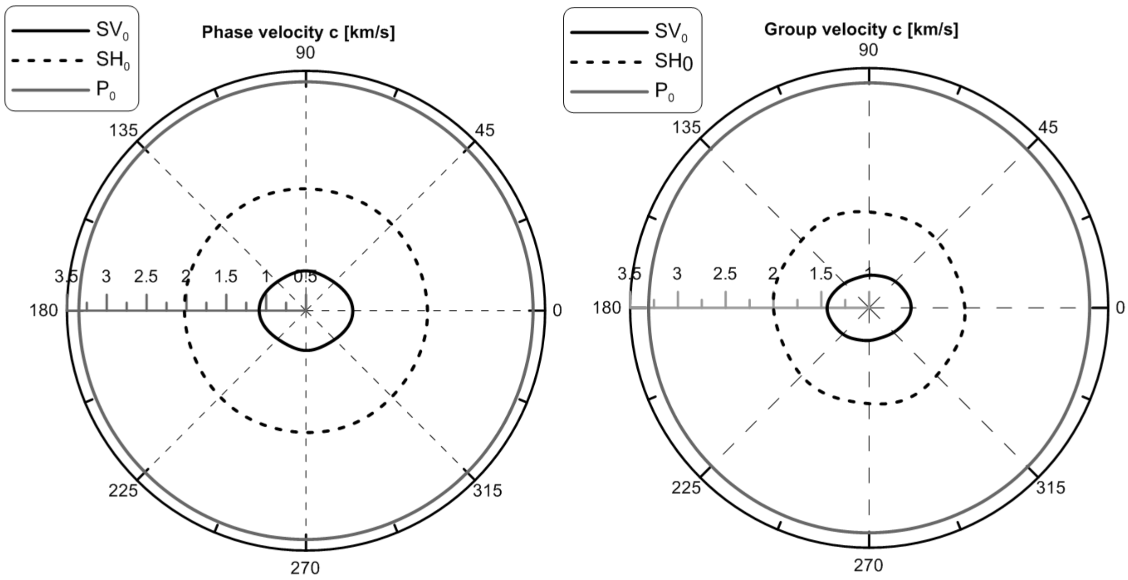


Fig. 6. Phase and group velocities with respect to waves angle propagation φ for composite [0°, 90°, 45°, -45°, -45°, 45°, 90°, 0°]. Total thickness of layer t_c=2 mm. Fixed frequency f = 250 kHz

4. Conclusions

It should be stressed here that the applied here stiffness matrix method is an effective tool for determining dispersion curves for any arbitrary multi-layered composite materials. It is relatively simple and easy to use in comparison with, for example, global matrix method. Generally, the shape and the number of elastic wave modes, which are present in the investigated range of frequency and phase velocity, strictly depends on the mechanical properties of the whole composite structure as well as on the waves propagation angle φ. For the angle φ different from 0° and 90°, the number of higher modes is the largest. Qualitatively, the behaviour of the fundamental modes is similar in all investigated cases. For the low frequency, the highest phase and group velocity has always the symmetric mode P_0.

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