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SOLUTIONS OF RECURRENCES WITH VARIABLE COEFFICIENTS FOR SLIDE BEARING WEAR DETERMINATION

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Abstract

The numerous methods of numerical calculations occurring in power-train tribology and transport concerning wear bearing determination problems demand the more and more information referring the slide bearing wear anticipation in succeeding years of machine operations. Therefore this paper presents the methods of solutions of some specific class of ordinary non-homogeneous recurrence equations of second and higher order with variable coefficients occurring in hydrodynamic theory of bearing wear problems.

Contrary to linear recurrence equations with constant coefficients, linear recurrence equations with variable coefficients rarely have analytical solutions.. Numerical solutions of such equations are always practicable. In numerous analytical methods of solutions of linear recurrence equations with variable coefficients there are usually three research directions. The first of them depend upon the successive determination of the linear independent particular solutions of the considered recurrence equation. The second direction to be characterized by the reduction of the order of recurrence equation to obtain an always solved, first order recurrence equation. The third direction of solutions of recurrence equations with variable coefficients, contains the methods of analytical solutions by means of a summation factor. The majority of the general methods of analytical solutions of linear recurrence equations with variable coefficients constitute an adaptation of the methods applied in solutions of suitable differential equations

In final conclusions the application of presented theory in this paper contains the the examples referring the wear values determination of HDD bearing system in the indicated period of operating time.

Keywords: *wear anticipation after exploitation, slide bearings, recurrence equation, variable coefficients*

1. Initial information about micro-bearing wear prognosis

The presented paper describes the stochastic calculation method of wear anticipation in HDD micro-bearing taking into account the dynamic behavior during the operating time.

The wear prognosis of two cooperating HDD micro-bearing surfaces during the operation time has very important meaning in contemporary software technological processes. Solution of presented problem can be possible on the basis of recently obtained measurements of micro-bearing wear during the first periods (may be month or years) of exploitation and regarding the parameters describing the random wear effects, completed by the dimensional standard deviation.

After experimental AFM measurements follows, that discrete wear values f_{n+2} of the sequence $\{f_n\}$ for $n = 1, 2, 3,\dots$ i.e. the wear values increases in mm^3 of HDD micro-bearing journal and sleeve equal to sum ($P_n f_{n+1}+ Q_n f_n$) of wear in two foregoing successive time units (may be months) where addends are multiplied by dimensionless variable stochastic wear coefficient P_n and Q_n plus some dimensional, variable stochastic values function of materials R_n. Variable coefficients P_n, Q_n, R_n depend on the obtained in experimental way dimensional wear standard deviation of micro-bearing material, the journal angular velocity and the frequencies of vibrations. Variable n is numbered by natural numbers 1, 2, 3,…

The dimensionless n dependent i.e. variable random parameters P_n, Q_n denote any coefficients which average the wear in two succeeding foregoing time units. Standard deviation dependent term R_n, and average term ($P_n f_{n+1}$+ $Q_n f_n$) of two foregoing wears, make up the sequence values of real wear in succeeding time unit. In this case wear of HDD micro-bearing can be described by the following recurrence equation [5–7]:

$$f^*_{n+2} = (P_n f^*_{n+1} + Q_n f^*_n) + R_n \quad \text{for} \quad n = 1, 2, 3, \ldots \, . \tag{1}$$

Recurrent equation (1) determines analytical formula $\{f^*_n\}$ presenting a sequence of wear values numbered for n = 1, 2, 3,… time units if we know dimensionless values P_n [1], Q_n [1] and dimensional value R_n [mm^3]. To solve mentioned problem it is necessary to add the boundary conditions. Hence we assume that in two first time units (may be month), the wear by virtue of experiments attains dimensional values W_1 [mm^3], W_2 [mm^3], $W_1 < W_2$.

2. Wear determination in power train tribology by means of the replacement of variables

The method presented in this intersection relays on the new variable introduction which enables us to reduce the order of the recurrence equation. The example 1 illustrates the above-mentioned method [1–4].

EXAMPLE 1

Determine general solution f_n (n = 1, 2, 3,…) i.e. the wear for the following recurrence equation:

$$a\,n\,f_{n+2} - (a\,n - b) f_{n+1} - b\,f_n = 0 \Leftrightarrow f_{n+2} = \left(1 - \frac{b}{a\,n}\right) f_{n+1} + \frac{b}{a\,n} f_n\,, \tag{2}$$

where positive constant values a, $b \neq 0$ are independent of n. Show the particular wear solution for known boundary conditions i.e. wear values $f_1 = W_1$, $f_2 = W_2$ in two successive time units n = 1 and n = 2.

SOLUTION OF EXAMPLE 1

Equation (2) can be described in the following form:

$$a\,n(f_{n+2} - f_{n+1}) + b(f_{n+1} - f_n) = 0\,. \tag{3}$$

We assume the following new variable:

$$y_n \equiv f_{n+1} - f_n \quad \text{for} \quad n = 1, 2, 3 \ldots \tag{4}$$

and hence from the formula (4) follows:

$$y_{n+1} \equiv f_{n+2} - f_{n+1}\,. \tag{5}$$

Putting dependences (4), (5) into recurrent equation (3) we obtain the following linear homogeneous first order recurrent equation with a variable coefficient:

$$a\,n\,y_{n+1} + b\,y_n = 0 \Rightarrow y_{n+1} + \frac{b}{a\,n} y_n = 0\,. \tag{6}$$

Following first order homogeneous recurrence equation with variable coefficient A_n:

$$y_{n+1} + A_n\,y_n = 0, \quad A_n \equiv \frac{b}{a\,n}\,, \tag{7}$$

has the general solution in the form:

$$y_n = C(-1)^{n-1} \prod_{j=1}^{n-1} \left(\frac{b}{a\,j}\right) = \frac{C}{(n-1)!} (-1)^{n-1} \left(\frac{b}{a}\right)^{n-1} \quad \text{for} \quad n = 2, 3, 4 \ldots, \tag{8a}$$

$$y_1 = C, \tag{8b}$$

where C denotes the first arbitrary constant.

Solution (8a) in presented form y_n is introduced into substitution (4) and hence we obtain the following, linear, non-homogeneous first order recurrence equation:

$$f_{n+1} - f_n = \frac{C}{(n-1)!}\left(-\frac{b}{a}\right)^{n-1} \Leftrightarrow f_{n+1} - f_n = B_n \quad \text{for} \quad n = 1,2,3,4,\ldots \;. \tag{9}$$

Abovementioned first order non-homogeneous recurrence equation with following variable free term B_n:

$$B_n = \frac{C}{(n-1)!}\left(-\frac{b}{a}\right)^{n-1} \quad \text{for} \quad n = 1,2,3,4,\ldots \;. \tag{10}$$

has the general solution in the form:

$$f_n = C_1 \cdot (-1)^{n-1} \cdot (-1)^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k-1} \cdot (-1)^{n-k-1} \cdot \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1}, \quad \text{for} \quad n = 2,3,4,\ldots, \tag{11a}$$

$$f_1 = C_1. \tag{11b}$$

After terms ordering in Eq. (11), we obtain:

$$f_n = C_1 + \sum_{k=1}^{n-1} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1}, \qquad \text{for} \quad n = 2,3,4,\ldots, \tag{12a}$$

$$f_1 = C_1. \tag{12b}$$

Symbol C_1 denotes second arbitrary constant in the presented solution (12) of recurrence equation (9). To determine the particular solution of recurrence equation (2) for boundary conditions $f_1 = W_1$ and $f_2 = W_2$ in places $n = 1$ and $n = 2$, we ought to note, that assumption (4) for $n = 1$ and by virtue of (8b) we obtain:

$$C = W_2 - W_1 = f_2 - f_1 = y_1, \quad C_1 = W_1. \tag{13}$$

We put constant C and C_1 from Eq. (13) into Eq. (12) then the particular solution of recurrence Eq. (2) has the final particular wear value solution in following form:

$$f_n = W_1 + (W_2 - W_1)\sum_{k=1}^{n-1} \frac{(-\delta)^{k-1}}{(k-1)!} \qquad \text{for} \quad n = 2,3,4,\ldots,\ \delta \equiv b/a, \tag{14a}$$

$$f_1 = W_1. \tag{14b}$$

By virtue of (14) follows, that for wear values $f_1 = W_1$, $f_2 = W_2$ in two first time units $n = 1$, $n = 2$, the wear values during the next time units attain the following values:

$$\begin{aligned}
f_3 &= W_1 + (W_2 - W_1)(1-\delta),\\
f_4 &= W_1 + (W_2 - W_1)\left[1 + \frac{1}{1!}(-\delta) + \frac{1}{2!}(-\delta)^2\right],\\
f_5 &= W_1 + (W_2 - W_1)\left[\frac{1}{1!}(-\delta) + \frac{1}{2!}(-\delta)^2 + \frac{1}{3!}(-\delta)^3\right],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
f_n &= W_1 + (W_2 - W_1)\left[1 + \frac{1}{1!}(-\delta) + \frac{1}{2!}(-\delta)^2 + \ldots. + \frac{1}{(n-1)!}(-\delta)^{n-1}\right]
\end{aligned} \tag{15a}$$

and during the infinity time units the wear attain the value:

$$f_{\infty} = W_1 + (W_2 - W_1)\exp(-\delta) = W_1(1-\mathrm{e}^{-\delta}) + W_2\mathrm{e}^{-\delta}, \quad 1-\mathrm{e}^{-\delta} > 0. \tag{15b}$$

The wear values process during the particular time units is convergent. The sum of wear values after finite $N < \infty$ and infinite for $N \to \infty$ time units i.e. is divergent:

$$F_1 = W_1 + W_2 \cdot 0, \quad F_2 = W_1 + W_2,$$

$$F_N \equiv W_1 \sum_{n=1}^{N} \left\{ 1 - \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} + W_2 \sum_{n=1}^{N} \left\{ \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\}, \qquad N = 3,4,...,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{16}$$

$$F_{\infty} \equiv W_1 \sum_{n=1}^{\infty} \left\{ 1 - \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} + W_2 \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} \to \infty.$$

The sums of wear F_N, F_{N+1}, F_{N+2}, … always increase after successive N, N+1, N+2, … time units. Now we show the proof that, general solution (12) satisfies recurrence Eq. (2).

PROOF

We show the following steps of the sequence presenting the general solution (12):

$$f_{n+1} = C_1 + \sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1}, \qquad \text{for} \quad n = 2,3,4,..., \tag{17}$$

$$f_{n+2} = C_1 + \sum_{k=1}^{n+1} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1}, \qquad \text{for} \quad n = 2,3,4,.... \tag{18}$$

From formulae (17), (18), the following expressions are true:

$$a\,n\,f_{n+2} = C_1\,a\,n + a\,n \sum_{k=1}^{n+1} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} = C_1\,a\,n + a\,n\left[\sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} + \frac{C}{n!}\left(-\frac{b}{a}\right)^{n} \right] =$$
$$= C_1\,a\,n + a\,n \sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} + \frac{a\,C}{(n-1)!}\left(-\frac{b}{a}\right)^{n}, \tag{19}$$

$$-(a\,n-b)\,f_{n+1} = -C_1\,a\,n + b\,C_1 - (a\,n-b)\sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} =$$
$$= -C_1\,a\,n + b\,C_1 - a\,n \sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} + \sum_{k=1}^{n} \frac{C\,b}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1}. \tag{20}$$

Substituting the r.h.s. (right hand side) of Eq. (19) and (20), and equality (12) into recurrence equation (2), we obtain:

$$C_1\,a\,n + a\,n \sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} + \frac{C\,a}{(n-1)!}\left(-\frac{b}{a}\right)^{n} - C_1 a\,n + b\,C_1 +$$
$$- a\,n \sum_{k=1}^{n} \frac{C}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} + \sum_{k=1}^{n} \frac{C\,b}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} - b\,C_1 - \sum_{k=1}^{n-1} \frac{C\,b}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1}. \tag{21}$$

After term reduction and term ordering in expression (21), we obtain finally:

$$\frac{a\,C}{(n-1)!}\left(-\frac{b}{a}\right)^{n} + \sum_{k=1}^{n} \frac{C\,b}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} - \sum_{k=1}^{n-1} \frac{C\,b}{(k-1)!}\left(-\frac{b}{a}\right)^{k-1} = 0. \tag{22}$$

Hence recurrence equation (2) is satisfied. This fact completes the PROOF.

3. Wear process determination in power train tribology by means of the summation factor

The l.h.s. (left hand side) of the following recurrence equation with coefficients A_n, B_n, C_n and free term D_n:

$$A_n f_{n+2} + B_n f_{n+1} + C_n f_n = D_n \Leftrightarrow f_{n+2} = P_n f_{n+1} + Q_n f_n + R_n , \qquad (23)$$

is the total sum or the total differential. In this case we can choose such coefficients L_n, M_n, that the following equality is true:

$$S^1_{\varepsilon\rho}(L_n f_{n+1} + M_n f_n) = f_{n+2} - P_n f_{n+1} - Q_n f_n = R_n . \qquad (24)$$

And non-homogeneous reduced equation has the form:

$$L_n f_{n+1} + M_n f_n = S^{-1}_{\varepsilon\rho}(R_n) + C(-\varepsilon\rho)^n , \qquad (25)$$

where C is the arbitrary summation constant.

If the equality (24) is not valid, we can find the summation factor U_n for equation (23). After multiplication of both sides of the equation (23) by this factor, we can always find such coefficients L^*_n, M^*_n that by virtue of the total sum, the following equality is true:

$$S^1_{\varepsilon\rho}(L^*_n f_{n+1} + M^*_n f_n) = U_n f_{n+2} - U_n P_n f_{n+1} - U_n Q_n f_n = U_n R_n . \qquad (26)$$

And in this case reduced non-homogeneous equation has the following first order recurrence form:

$$L^*_n f_{n+1} + M^*_n f_n = S^{-1}_{\varepsilon\rho}(U_n R_n) + C(-\varepsilon\rho)^n . \qquad (27)$$

Symbol $S^{-1}_{\varepsilon\rho}$ denotes reciprocal unified operator of summation with the basis $\varepsilon\rho$.

Reciprocal UOS regarding the unified operator of summation occurring in Eq.(27) is denoted by the following description:

$$S^{-1}_{\varepsilon\rho}(...). \qquad (28)$$

The reciprocal unified operator of summation will be defined in the following form:

$$S^{-1}_{\varepsilon\rho}(x_n) \equiv X_n, \quad \text{because} \quad S^{+1}_{\varepsilon\rho}(X_n) \equiv x_n , \qquad (29)$$

where X_n, x_n are the functions determined for the natural numbers n = 1, 2, 3,….

Reciprocal unified operator of summation is denoted by the upper index (−1) and is not always univocal.

EXAMPLE 2

Determine general solution f_n (n = 1, 2, 3,…) i.e. the wear for the following recurrence equation:

$$\left(1+\frac{1}{n}\right) f_{n+2} + \left(n+\frac{1}{n}\right) f_{n+1} - 2n\, f_n = d \Leftrightarrow f_{n+2} = -\frac{n^2+1}{n+1} f_{n+1} + \frac{2n^2}{n+1} f_n + \frac{d\,n}{n+1}, \qquad (30)$$

where $d \neq 0$ denotes an arbitrary coefficient independent of n. Show the particular wear solution for known boundary conditions i.e. wear values $f_1 = W_1, f_2 = W_2$ in two successive time units $n = 1$ and $n = 2$.

SOLUTION OF EXAMPLE 2

L.h.s. of recurrence equation (30) does not present the total sum. Symbol n is the summation factor, because by multiplying by n both sides of equation (30) we obtain:

$$(n+1) f_{n+2} + (n^2+1) f_{n+1} - 2n^2 f_n = d\, n , \qquad (31)$$

and the following equality is true:

$$S^{1}_{-2}(n\,f_{n+1}+n^{2}f_{n})=(n+1)f_{n+2}+(n+1)^{2}f_{n+1}-2n\,f_{n+1}-2n^{2}f_{n}= \\ =(n+1)f_{n+2}+(n^{2}+1)f_{n+1}-2n^{2}f_{n}. \tag{32}$$

Hence equation (31) can be written in the following form:

$$S^{1}_{-2}(n\,f_{n+1}+n^{2}f_{n})=d\,n\,. \tag{33}$$

When a reciprocal operator is imposed on the both sides of equation (33), we obtain the following:

$$n\,f_{n+1}+n^{2}f_{n}=S^{-1}_{-2}(d\,n)\,. \tag{34}$$

By virtue of reciprocal operator properties we obtain:

$$S^{-1}_{-2}(d\,n)=d\,n\,J-d\,J^{2}+C\cdot 2^{n},\quad J=\frac{1}{1-2}=-1\,. \tag{35}$$

Symbol C denotes the first summation constant. Equation (34) has the form:

$$n\,f_{n+1}+n^{2}f_{n}=-d\,n-d+C\cdot 2^{n}\,. \tag{36}$$

Dividing both sides of Eq. (36) by n, we obtain:

$$f_{n+1}+n\,f_{n}=-d\left(1+\frac{1}{n}\right)+C\cdot\frac{2^{n}}{n}\cdot \tag{37}$$

The general solution of recurrence equation (37), has the following form:

$$f_{n}=(-1)^{n-1}\cdot\prod_{j=1}^{n-1}j\left\{C_{1}+\sum_{k=1}^{n-1}\left[\frac{\frac{C}{k}\cdot 2^{k}-d\left(1+\frac{1}{k}\right)}{(-1)^{k}\prod_{s=1}^{k}s}\right]\right\}\quad\text{for}\quad n=2,\,3,\ldots, \tag{38a}$$

$$f_{1}=C_{1}\,. \tag{38b}$$

Symbol C_1 denotes the second arbitrary constant of summation. After transformations, of formula (38), we obtain:

$$f_{n}=(-1)^{n-1}\cdot(n-1)!\left\{C_{1}+\sum_{k=1}^{n-1}\left[(-1)^{k}\,\frac{\frac{C}{k}\cdot 2^{k}-d\left(1+\frac{1}{k}\right)}{k!}\right]\right\}\quad\text{for}\quad n=2,\,3,\ldots, \tag{39a}$$

$$f_{1}=C_{1}\,. \tag{39b}$$

On solution (39a, b) we impose boundary condition $f_1 = W_1$ for $n = 1$ and $f_2 = W_2$ in place $n = 2$, hence we obtain the following summation constants:

$$C_{1}=W_{1},\,C=d+\frac{1}{2}(W_{1}+W_{2})\,. \tag{40}$$

In presented formula (39) we can now show a linear combination of two linear independent particular solutions of the homogeneous equation plus the particular solution of non-homogeneous recurrence accordingly with Eq. (30). Abovementioned linear combination of particular solutions is written in following form:

$$f_n = W_1 f_n^{[1]} + W_2 f_n^{[2]} + f_n^{[b]}, \quad n = 1,2,3,..., \tag{41a}$$

$$f_n^{[1]} = (-1)^{n-1}(n-1)!\left[1 + \sum_{k=1}^{n-1}(-1)^k \frac{2^{k-1}}{k\,k!}\right], \quad f_1^{[1]} = 1,\ n = 2,3,..., \tag{41b}$$

$$f_n^{[2]} = (-1)^{n-1}(n-1)!\sum_{k=1}^{n-1}(-1)^k \frac{2^{k-1}}{k\,k!}, \quad f_1^{[2]} = 0,\ n = 2,3,..., \tag{41c}$$

$$f_n^{[b]} = d(-1)^{n-1}(n-1)!\sum_{k=1}^{n-1}\frac{(-1)^k}{kk!}(1+k-2^k),\ f_1^{[b]} = 0,\ n = 2,3,...k = 1,\ 2,...,n-1\,. \tag{41d}$$

4. Conclusions

A) The application of the presented theory contains the analytical methods of solutions of second order recurrent equations with variable coefficients referring the wear values determination of bearing system in the considered period of the operating time.

B) The recurrences equations determining the wear values in power train tribology are solved by means of the replacement of variables and summation factor.

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