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RECURRENT NETS FOR TRIBOLOGY SOLUTIONS

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Abstract

The contemporary problems of numerical calculations ocurring in powertrain tribology and transport problems demand the more and more exactness for obtained results.Moreover in performed calculations very important is the convergence, stability and reliability of the gained numerical values. The main scientific topic of the presented paper concerns the method of the determination of the optimum net for numerical calculations of partial difference and recurrence equations. The abovementioned optimum difference and recurrence method is referring to the stability of obtained particular and general numerical solutions and assures the convergence process of obtained calculation values. The Unit Net Region (UNR) was assumed at first for Laplace Operator. The optimum of the nod geometry localization was examined at first for UNR. The optimization index is defined and derived for UNR to determine the most useful net among the various geometries of the nods localization during the difference methods performances of partial recurrence numerical calculations. In the next considerations had been proved the corollary, where taking into account the optimum UNR, we can create optimum nets for other numerous partial difference and recurrence equations in discrete spaces. For example the numerous calculation results of presented optimum net for recurrent calculations are applied for numerical solutions of a Reynolds partial recurrence equation with variable coefficients in curvilinear orthogonal coordinates for curvilinear boundary conditions, and for other numerical problems occurring in applied and hydrodynamics.

Keywords: *pressure changes in gap height direction, HDD micro-bearings, viscoelastic lubrication*

1. Introduction

The main scientific topic of the presented paper concerns the method of the determination and calculations of the general and particular numerical solutions of a partial recurrence equation with variable coefficients in curvilinear orthogonal coordinates for curvilinear boundary conditions. The paper show an indication the new applications of the abovementioned solutions in powertrain, tribology and transport problems.A general space of discrete solution was derived and determined for various orthogonal coordinates. From the mathematical point, the presented method of the solution of Poisson or modified Reynolds equation leads this problem to resolving the partial recurrence non-homogeneous, linear equation of the second order with variable coefficients.

Furthermore, the topology of solutions presented in this paper on the space of the discrete values for partial recurrence equations and the topology on the space of continuous functions as the solutions of partial difference equations makes it possible to select an optimum numerical methods. Hence in this paper was being considered optimum scheme of recurrent performances and optimum net for recurrent calculations from among many methods in relation to the stability of solutions, a convergence of obtaining calculations as well as the duration time of calculations in the scope of various orthogonal and non-orthogonal coordinates which describe the geometries of boundary conditions.

2. The net of differences nods for the function of two variables

In this intersection, we will assume an optimum difference scheme which leads to the recurrent equations for Poisson equation (1a) with Dirichlet boundary conditions (1b) [1-3, 5-7]:

$$\Delta F \equiv \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = G(x, y), \tag{1a}$$

$$F(x, y) = \Phi(x, y), \quad (x, y) \in \partial\Omega \tag{1b}$$

in square region A×A; Ω:$\{(x, y)$ for $0 \le x, y \le A\}$.

Let function $F(x, y)$ of two variables x, y in rectangular coordinates (x, y) be determined on region Ω defined in (1b). The region is covered over the net created by the straight lines parallel to the axis x and y in the sub-intervals by distance h and k, respectively. Such a division can be defined in the following form:

$$x_i = i\,h,\; y_i = j\,k \quad \text{for } i, j = 0, 1, 2, \ldots, n. \tag{2}$$

Sub-intervals (step) h, k denote the distances between parallel straight lines, located in the division points and that are perpendicular to the axes x or y. The cut points of the above mentioned lines form the nods inside region A×A presented in Fig. 1:

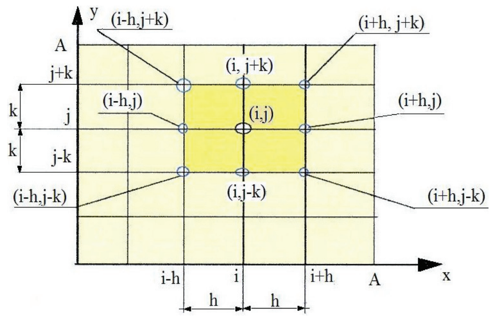

Fig. 1. Region A×A covered by the net

3. Laplace Operator in a recurrence form

To derive the Laplace operator in the difference form we remark that the properties of Taylor series are important for the application for the transformation to the recurrence form. Function $F(x, y)$ of two variables is expanded in the Taylor series [4–5] in the neighborhood of point $(x + h, y + k)$:

$$F(x+h, y+k) = F(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)F(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 F(x, y) + \ldots +$$
$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n F(x + \Theta_x h, y + \Theta_x k), \tag{2}$$

where $0 < \Theta_x, \Theta_y < 1$. We assume the following notation of the derived functions:

$$\frac{\partial F}{\partial x} \equiv F_x, \quad \frac{\partial F}{\partial y} \equiv F_y, \quad \frac{\partial^2 F}{\partial x^2} \equiv F_{xx}, \quad \frac{\partial^2 F}{\partial y^2} \equiv F_{yy}, \quad \frac{\partial^2 F}{\partial x \partial y} \equiv F_{xy}, \;\ldots. \tag{3}$$

Making use of Taylora formula (2) and taking into account notations (3), we lead function $F(x, y)$ in neighborhood (i, j) into the following difference form:

$$F_{i+h,\,j+k} = F_{ij} + h(F_x)_{ij} + k(F_y)_{ij} + 0.5h^2(F_{xx})_{ij} + hk(F_{xy})_{ij} + 0.5k^2(F_{yy})_{ij} + \mathrm{O}(h^3, k^3), \tag{4a}$$

$$\Theta(h^3, k^3) \equiv \frac{1}{6}(h^3 F_{xxx} + 3h^2 k F_{xxy} + 3h k^2 F_{xyy} + k^3 F_{yyy})_{ij} +$$
$$+\frac{1}{24}(h^4 F_{xxxx} + 4h^3 k\, F_{xxxy} + 6h^2 k^2 F_{xxyy} + 4h k^3 F_{xyyy} + k^4 F_{yyyy})_{ij} + \ldots. \tag{4b}$$

Equation (4a) satisfies the following difference form of the LO (Laplace Operator):

$$0.5(h^2F_{xx}+k^2F_{yy})_{ij}=F_{i+h,j+k}-F_{ij}-h(F_x)_{ij}-k(F_y)_{ij}-hk(F_{xy})_{ij}-\mathrm{O}(h^3,k^3)\,. \quad (5)$$

The identity (5) is visible if we put (4a) into r.h.s. (right hand side) of equation (5).

4. First and second step of Unit Net Region (UNR) approximation Δ_1, Δ_2

Into Eq.(5) we put the two sets of the following four ordered pair values h, k [4-6]:

$$(h=1,\ k=0),(h=-1,\ k=0),(h=0,\ k=+1),(h=0,\ k=-1)\,, \quad (6)$$

$$(h=-1,\ k=-1),(h=1,\ k=-1),(h=-1,\ k=+1),(h=1,\ k=-1)\,. \quad (7)$$

Each from the two abovementioned sets of four pair values (6) and (7) substituted into Eq.(5) gives its another form of equations. We add such four obtained equations mutually. Thus for (6), (7) we obtain the following recurrence UNR forms:

$$\Delta_1F_{ij}\equiv h^2(F_{xx}+F_{yy})_{ij}=F_{i+1,j}+F_{i-1,j}+F_{i,j+1}+F_{i,j-1}-4F_{i,j}+\mathrm{O}_1(1), \quad (8)$$

$$\text{for } \mathrm{O}_1(1)\equiv-\Theta(F_{xxxx}+F_{yyyy})_{ij}\,,$$

$$\Delta_2F_{ij}\equiv 2h^2(F_{xx}+F_{yy})_{ij}=F_{i-1,j-1}+F_{i+1,j-1}+F_{i+1,j+1}+F_{i-1,j+1}-4F_{i,j}+\mathrm{O}_2, \quad (9)$$

$$\text{for } \mathrm{O}_2\equiv-2\mathrm{O}_1(F_{xxyy})_{ij}\,.$$

where $\Theta = 1/12$. Dimension conformability demands to multiply the l.h.s. (left hand side) of Eqs. (8) and (9) by h^2 for $h = k$. Fig. 2 presents the UNR net of nods for the difference form (8) and (9). For respective nods the following coefficient values are ascribed: 1, 1, 1, 1, –4 and 1, 1, 1, 1, –4. Such coefficients occur by the appropriately indicated terms of recurrent formula (8) and (9), respectively.

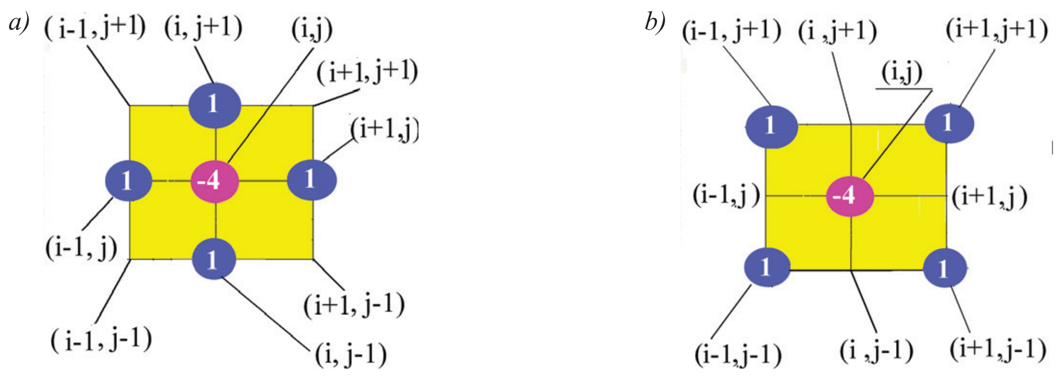

Fig. 2. The net with nods for a) first and b) second case of approximation difference form for UNRΔ_1 and Δ_2

In the net presented in Fig. 2, the internal nod is fourfold and the external nods are single. The fourfold nod denotes that at the term $F_{i,j}$ in recurrent equations (8), (9) coefficient 4 occurs. The external nods and the internal nod form the geometry element of the UNR.

5. Third step of Unit Net Region (UNR) approximation Δ_3

The third case of the approximation difference form for UNR will be defined by the union of two foregoing forms and is described by the following formula [4–6]:

$$\Delta_3F_{ij}\equiv(a\,\Delta_1+b\,\Delta_2)F_{ij} \qquad \text{for} \quad a=1 \text{ and } b=1\,. \quad (10)$$

Putting approximation forms (8)+(9) into (10), we obtain:

$$\Delta_3 F_{ij} \equiv 3h^2 (F_{xx} + F_{yy})_{ij} = F_{i-1,j-1} + F_{i,j-1} + F_{i+1,j-1} + F_{i-1,j} + F_{i+1,j} + \\ + F_{i+1,j+1} + F_{i-1,j+1} + F_{i,j+1} - 8F_{i,j} + \mathrm{O}_3, \quad \text{for} \quad \mathrm{O}_3 \equiv \mathrm{O}_1 + \mathrm{O}_2. \tag{11}$$

Figure 3 presents the net of nods for difference form (11). To respective nods the coefficient values are ascribed: 1, 1, 1, 1, 1, 1, 1, 1, –8 occurring by the appropriately indicated terms of recurrent formula (11).

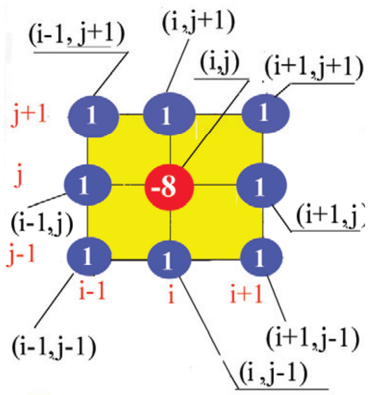

Fig. 3. The net with nods for the third case of the approximation difference form for UNR Δ3

In the presented net in Fig. 3 the internal nod is eightfold and the external nods are single. The eightfold nod denotes that in recurrent equations (10) at term $F_{i,j}$ the coefficient 8 occurs. The external nods and the internal nod form the element of the UNR.

6. Fourth step of Unit Net Region (UNR) approximation Δ4

The fourth case of the approximation difference form for UNR will be defined by the combination of two foregoing forms using the following formula [4]:

$$\Delta_4 F_{ij} \equiv (a\Delta_1 + b\Delta_2) F_{ij} \qquad \text{for} \quad a = 4 \text{ and } b = 1. \tag{12}$$

We put approximation forms (8)+(9) in Eq. (12). Thus we obtain:

$$\Delta_4 F_{ij} \equiv 6h^2 (F_{xx} + F_{yy})_{ij} = 4(F_{i-1,j} + F_{i+1,j} + F_{i,j-1} + F_{i,j+1}) + F_{i-1,j-1} + F_{i+1,j-1} \\ + F_{i+1,j+1} + F_{i-1,j+1} - 20F_{i,j} + \mathrm{O}_4 \qquad \text{for} \quad \mathrm{O}_4 \equiv 4\mathrm{O}_1 + \mathrm{O}_2. \tag{13}$$

Figure 4 presents the net of nods for difference form (13).

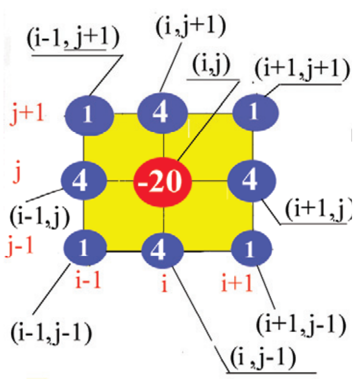

Fig. 4. The net with nods for fourth case of approximation difference form for UNR Δ4

The coefficient values are ascribed to respective nods: 1, 4, 1, 4, 1, 4, 1, 4, −20 occurring by the appropriately indicated terms of recurrent formula (13). In the net presented in Fig. 4 the internal centre nod is twenty fold and the four corner external nods are single, whereas the remaining external nods are fourfold. The twenty fold centre nod denotes that at term $F_{i,j}$ coefficient 20 occurs in recurrent equations (13). The external nods and the internal nod form the element of the UNR.

7. Index of optimization for UNR approximation

Now we show the way of the stability evaluation of numerical solutions for the individual difference approximations of UNR. We define the following optimization index:

$$c \equiv \frac{M}{\sum_{r=1}^{R} W_r}, \qquad (14)$$

where r – the index of movable nod in the element of net ($r = 1, 2,.., R$), R – the quantity of possible movable nods in an element of the net, W_r – the multiplicity of coefficients in the r-th movable nod during the conversion step from one to the next position in the numerical calculations of the recurrence equation, M – the multiplicity of covering of non zero coefficients in all the nods during the conversion step from one position to the next position in the numerical calculations of the recurrence equation.

Index c denotes ratio of the cover quantity of non-zero coefficients to the one movable coefficient during the conversion from one position to the next position of the net element in the numerical calculations. The stability of the calculation process increases if index c increases.

8. Optimization index determination for UNR conversions

Now we calculate optimization index c for the above mentioned recurrence UNR forms of approximation [5–7].

Figure 5 illustrates the conversion of the net element with movable nods from one position to the next position during the numerical process in the first case of the approximation difference form for UNR Δ_1.

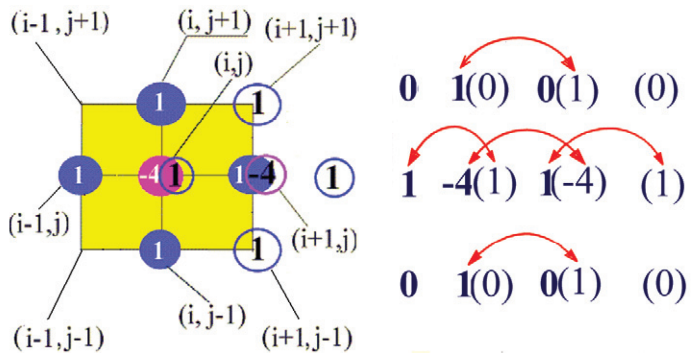

Fig. 5. The covering geometry of nods during the conversion of the movable net element from one position to the next position in numerical calculations for the first case of the approximation recurrence UNR Δ_1 form

In the above mentioned case we have: $R = 5$, $M = 1 + 1 = 2$, $\Sigma W_r = 1 + 1 + 4 + 1 + 1 = 8$ and $c = 2/8 = 0.25$.

Figure 6 illustrates the conversion of the net element with movable nods from one position to the next position during the numerical process in the second case of the approximation recurrence form for UNR Δ_2.

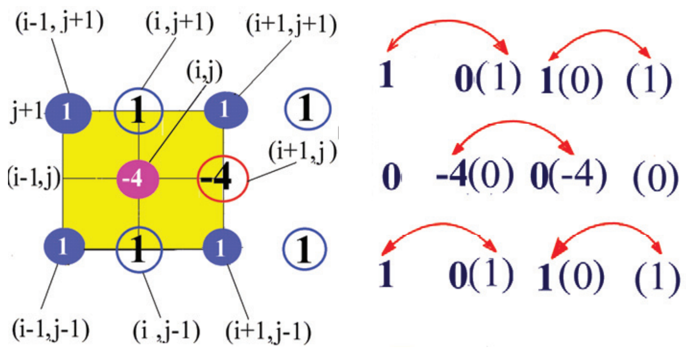

Fig. 6.The covering geometry of nods during the conversion of the movable net element from one position to the next position in numerical calculations for the second case of approximation recurrence UNR Δ_2 form

In the above mentioned case we have: $R = 5$, $M = 0$, $\Sigma W_r = 1 + 1 + 4 + 1 + 1 = 8$ and $c = 0/8 = 0$.

Figure 7 illustrates the conversion of the net element with movable nods from one position to the next position during the numerical process in the third case of the approximation recurrence form for UNR Δ_3.

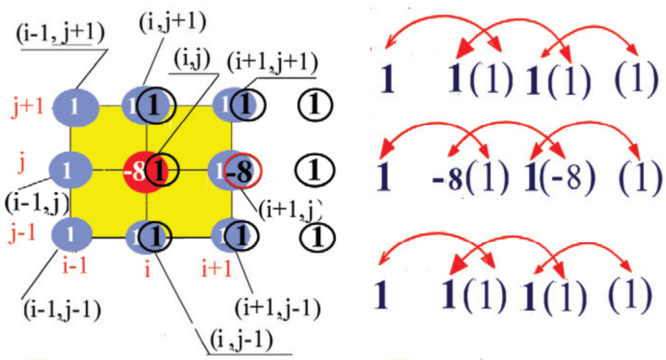

Fig. 7. The covering geometry of nods during the conversion of the movable net element from one position to the next position in numerical calculations for the third case of approximation recurrence UNRΔ_3 form

In the above mentioned case we have: $R = 9$, $M = 1 + 1 + 1 + 1 + 1 + 1 = 6$, $\Sigma W_r = 1 + 1 + 1 + 1 + + 8 + 1 + 1 + 1 + 1 = 16$, and optimization index equals $c = 6/16 = 0.375$.

Fig. 8 illustrates the conversion of the net element with movable nods from one position to the next position during the numerical process in the fourth case of the approximation recurrence form for UNR Δ_4.

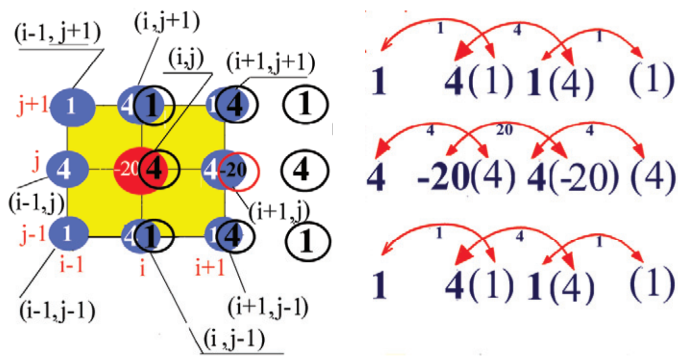

Fig. 8. The covering geometry of nods during the conversion of the movable net element from one position to the next position in numerical calculations for the fourth case of the approximation recurrence UNRΔ_4 form

In mentioned case we have: $R = 9$, $M = 1 + 1 + 4 + 4 + 1 + 1 = 12$, $\Sigma W_r = 1 + 4 + 1 + 4 + 20 + 4 + + 1 + 4 + 1 = 40$, $c = 12/40 = 0.300$.

COROLLARY 1: In the succeeding cases presented for numerical procedures we have the following values of optimization coefficients: 0.25, 0.00, 0.375, 0.300. In numerical calculations the third case is the most stable because optimization index c is the largest and has value 0.375.

9. The method of solution for the first case of UNR approximation

The method of the first case of approximation UNR Δ_1 will be applied for the transformation of a partial differential Poisson equation to the partial recurrence equation in region Ω. At first we multiply both sides of classical Poisson equation (1a) by factor $-h^2$ and next the obtained l.h.s. of (1a) is replaced by the expression $-\Delta_1 F_{ij}$ from the r.h.s. of Eq. (8). Neglecting the negligibly small terms O_1 we obtain the following linear system of equations:

$$4F_{i,j} - F_{i-1,j} - F_{i+1,j} - F_{i,j-1} - F_{i,j+1} = -h^2 G_{i,j}, \tag{15}$$

for internal nods $i, j = 1, 2,\ldots, N-1$; whereas for $4N$ external nods we have:

$$F_{k,j} = \Phi_{k,j},\ F_{i,k} = \Phi_{i,k},\ for\ k = 0, N;\ i, j = 0,1,\ldots,N;\ G_{i,j} = G(x_i, y_j),\ \Phi_{i,j} = \Phi(x_i, y_j). \tag{16}$$

The system of Eq.(15) has $(N-1)^2$ unknown i.e. the values of function F in the internal nods of region Ω. For the nods laying on the boundary of region Ω we impose boundary conditions (16), where the values of function Φ are known. Translating indexes i, j in Eq. (15) we obtain the partial, linear, non-homogeneous recurrence equations of the second order in the following form:

$$+F_{i+2,j+1} - 4F_{i+1,j+1} + F_{i+1,j+2} + F_{i+1,j} + F_{i,j+1} = +h^2 G_{i+1,j+1}. \tag{17}$$

The $(N-1)^2$ unknown values occurring in the set of Eq. (15) and their r.h.s. are described in the following matrix forms [1–4]:

$$\mathbf{F} = [F_{1,1}, F_{2,1},\ldots, F_{N-1,1},\ldots, F_{1,N-1}, F_{2,N-1},\ldots, F_{N-1,N-1}]^T, \tag{18}$$

$$\mathbf{B} = -h^2 [G_{1,1}, G_{2,1},\ldots, G_{N-1,1},\ldots, G_{1,N-1}, G_{2,N-1},\ldots, G_{N-1,N-1}]^T. \tag{19}$$

System of equations (15) can be written in the following matrix form [4]:

$$\mathbf{A}\,\mathbf{F} = \mathbf{B}, \tag{20}$$

where matrix **A** presents the following three-diagonal block matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & 0 & 0 & \ldots & 0 & 0 & 0 & 0. \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{A_{23}} & 0 & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & . & \ldots & . & . & . & . \\ . & . & . & . & \ldots & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & 0 & \mathbf{A_{N-2,N-3}} & \mathbf{A_{N-2,N-2}} & \mathbf{A_{N-2,N-1}} \\ 0 & 0 & 0 & 0. & \ldots. & 0 & 0 & \mathbf{A_{N-1,N-2}} & \mathbf{A_{N-1,N-1}} \end{bmatrix}. \tag{21}$$

Blocks $\mathbf{A}_{ij}$ denote the minors of degree $N-1$ with constant elements in each row whereas $\mathbf{A}_{ij} = \mathbf{E}$ (**E** unit matrix) for $i \neq j$ and:

$$\mathbf{A_{ii}} = \begin{bmatrix} 4 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0. \\ -1 & 4 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & . & \ldots & . & . & . & . \\ . & . & . & . & \ldots & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \ldots. & 0 & 0 & -1 & 4 \end{bmatrix}, \tag{22}$$

for $i = 1, 2, \ldots, N$- 1.

The methods of the immediate solutions of the system of equations (15) or (20) require using a computer with large operating memory if the number of the division of the calculated region attains value N: 5–100. Because matrix **A** is symmetric and is determined positive, during the calculation procedure we use iteration methods.

10. The solution method of solving for other approximation cases

We can apply the third LNRΔ_3 or fourth LNRΔ_4 case of approximation for the transformation of the partial differential Poisson equation or Reynolds equation to the partial recurrence equation in region Ω.

In such cases we multiply both sides of classical Poisson equation by factor $-3h^2$, or $-6h^2$ and next the obtained l.h.s. of this equation is replaced by expression $-\Delta_3 F_{ij}$ or $-\Delta_4 F_{ij}$ from the r.h.s. of Eq.(11) or Eq.(13), respectively. Neglecting the negligibly small terms O_2 or O_3 or O_4 we obtain the proper partial recurrence equations and the corresponding sets of linear system of algebraic equations [5–7].

11. Conclusions

In this paper have been presented the various geometries of the nets with calculation nods for difference methods of partial recurrence numerical solutions.The dynamic changes and conversions of particulat nets occurring during the numerical calculation process have been considered. As the result of performed researches, the optimum net regard to the stability and convergences of obtained numerical calculations of partial difference and recurrence equations have been discussed.

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