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LINEARIZATION OF THE SHIP EQUATIONS OF MOTION

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Abstract

In real systems are non-linear mathematical description. The exact solution can not be determined, and then look for approximate methods. Important is the type of nonlinearity, solutions and error method approximation. Linearization is an essential part of creating a model of the selected process. Ship resistance is a function of power with exponent two and higher. Model motion of the ship must have a solution in terms of maneuverability speed and speed of the sea. The solution must be well reproduce the actual path of the transition and the transition time of the ship. Nonlinear solution method determines the accuracy of the answers. Has presented the revised approach to solve the nonlinear differential equation of parabolic function. Linearization has been made in the selected range, and not where you want it to work and solve the error estimate. Range of solutions selected by external priorities adopted. Before the solution is estimated response error. The error value determines whether the selected interval will apply. If the problem solution is unacceptable, it will increase the accuracy of the result of the narrow scope of the work. The new scope of work should also be reassessed a solution error. This type of approach correlates with fuzzy logic, where we use the value of the Boolean variable with the function of belonging. The combination of classical methods of solving differential equations of the theory of fuzzy sets can bring new benefits. Such a solution must have the function of the accuracy of the answers. The linearization method meets this requirement.

Keywords: *transport, Mechanical Engineering, Maritime Engineering, non-linear differential equations, linearization, the resistance of the hull, error solutions*

Introduction

The effect of the resistance occurs very common in the technique. We know the resistance of electrical, hydraulic, pneumatic, mechanical and others. Each of the resistances may be linear, parabolic or any exponent. For example, a ship on the water resistance describes the parabolic function, but depending on sea conditions, this function may be higher or lower exponent. As the solution of nonlinear equations is generally not known, so it is determined approximate methods. Presented below linearization method can estimate the accuracy of the solution.

1. Linearization

The differential equation of the form:

$$\dot{y} + R(y) = F(t), \qquad y(t_0 = \cdots) = y_o, \tag{1.1}$$

is frequently used in technique to determine the dynamics of the system. For example, the motion of a material point of unit weight, caused by the force F(t), in an environment resisting R as a function of velocity **y** is described by the equation (1.1).

Arc of the curve with equation:

$$R = dy^l, \quad where \quad l \neq 0 \; il \neq 1, \tag{1.2}$$

in the interval $y_a \leq y \leq y_b$ will be replaced by its tangent **St** and secant **Sc** parallel to the tangent, as shown in Fig. 1.1.

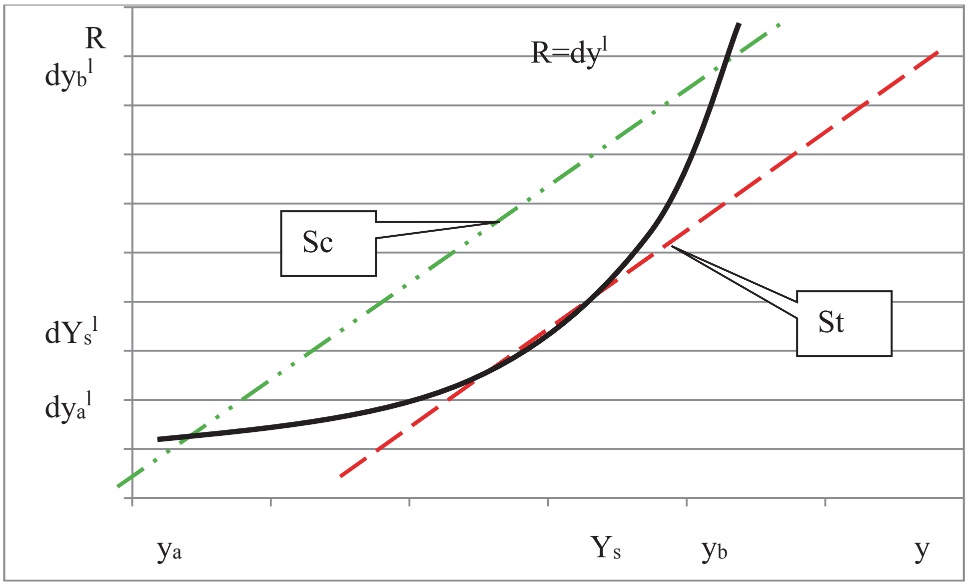

Fig. 1.1. Selected arc curve linearization

Because the secant

$$Sc = ay + b, \tag{1.3}$$

where:

$$a = \frac{dy_b^l - dy_a^l}{y_b - y_a}, \tag{1.4}$$

passes through the points A(y_a, dy_a^l) and B(y_b, dy_b^l), hence:

$$b = d(y_a^l - ay_a) \;\; or \;\; b = d(y_b^l - ay_b). \tag{1.5}$$

Tangent:

$$St = aY + B \tag{1.6}$$

has the curve(1.1) a common point C(Y_s, dY_s^l). At this point the derivative of the curve (1.1) and tangential (1.6) have the same value, thus:

$$a = dlY_s^{l-1} \;\; and \;\; Y_s = \left(\frac{a}{l}\right)^{\frac{1}{l-1}}. \tag{1.7}$$

Because:

$$dY_s^l = aY_s + B, \tag{1.8}$$

from here:

$$B = dY_s^l - aY_s = dY_s^l - dlY_s^{l-1} \cdot Y_s = dY_s^l(1 - l). \tag{1.9}$$

Equation (1.1) after replacing in the arc of the curve (1.2) in the range $y_a \le y \le y_b$the secant (1.3) and tangential (1.6) is limited by two equations:

$$\dot{y} = F(t) - ay - b, \qquad y \in < a, b > , \qquad y(t_0) = y_o, \tag{1.10}$$

$$\dot{Y} = F(t) - aY - B, \qquad Y \in < a, b > , \qquad Y(t_0) = y_o, \tag{1.11}$$

on the solutions:

$$y = de^{-at} + e^{-at} \int [F(t)e^{at}]dt - \frac{b}{a}, \tag{1.12}$$

$$Y = De^{-at} + e^{-at} \int [F(t)e^{at}]dt - \frac{B}{a}. \tag{1.13}$$

For the initial condition$y(t_0) = Y(t_0) = y_o$the integral have the form:

$$d = \left[y(t_0) + \frac{b}{a}\right] e^{at_0} - \int [F(t)e^{at}]dt|_{t=t_0}, \tag{1.14}$$

$$D = \left[y(t_0) + \frac{B}{a}\right] e^{at_0} - \int [F(t)e^{at}]dt|_{t=t_0}. \tag{1.15}$$

The difference solutions to equations (1.12) and (1.13) are:

$$2\Delta = Y - y = (D-d)e^{-at_0} + \frac{b-B}{a}. \tag{1.16}$$

After taking into account relationships (1.14) i (1.15) the equation takes the form:

$$2\Delta = Y - y = \frac{b-B}{a}\left(1 - e^{a(t_0 - t)}\right), \tag{1.17}$$

where:

$$Y_a = y_a < y < y_b = Y_b.$$

Unknown exact solution of equation (1.1) in the range of $y_a \le y \le y_b$ is between approximate solutions (1.12) and (1.13). The difference between the solutions (1.17), or the approximate solution error decreases with decreasing ratio (b-B)/a. This error can be determined without solving equations (1.10) and (1.11), using the values of the coefficients **a** and **b** secant (1.3) and the coefficient **B** of the tangent (1.6). Shortening the interval $y_a \le y \le y_b$or spreading it on the sub-intervals reduces the value of the difference (1.17). Now, however, be calculated constant **d** and **D** for each of the subintervals.

2. Selected examples of calculation

2.1. Example 1

Find an approximate solution of the equation:

$$\dot{y} + y^2 = t^2 + 1 \quad for \quad y(t=0) = 4, \tag{2.1}$$

in the range:

$$1 \le y \le 4. \tag{2.2}$$

Arc of the curve $R = y^2$ in the range $y_a = 1 \le y \le y_b = 4$ from equation (1.3) and (1.6) reduce:

secant $$Sc = 5y - 4 \tag{2.3}$$

and tangent $$St = 5Y - 6.25. \tag{2.4}$$

Equation (2.1), by replacing a arc curve by secant legs (2.3) and the tangent (2.4) is limited by two approximate equations in the range (2.2):

$$\dot{y} = t^2 + 1 - (5y - 4), \tag{2.5}$$

$$\dot{Y} = t^2 + 1 - (5Y - 6.25). \tag{2.6}$$

The solutions of equations (2.5) and (2.6) are the following:

$$y = de^{-5t} + \frac{1}{5}t\left(t - \frac{2}{5}\right) + \frac{2}{125} + \frac{1}{5} + \frac{4}{5}, \tag{2.7}$$

where y(t=0)=4 and d=2.984,

$$Y = De^{-5t} + \frac{1}{5}t\left(t - \frac{2}{5}\right) + \frac{2}{125} + \frac{1}{5} + \frac{6.25}{5}, \tag{2.8}$$

where D=2.534.

Table 2.1 shows the values **y** by the formula (2.7), **Y** by (2.8), **2Δ** according to (1.17), $\mathbf{y_{śr}}$ = y + Δ, the exact solution $\mathbf{y_d}$ according to the relationship:

$$y_d = t + \frac{e^{-t^2}}{C + \int e^{-t^2} dt} = \mathrm{t} + \frac{4e^{-t^2}}{1 + 2\sqrt{\pi}\,\mathrm{Erf}(t)}\,, \qquad C = 0.25, \tag{2.9}$$

and the value of$\beta = \frac{\Delta}{y_{śr}} \cdot 100\%$.

The results of calculations contained in Tab. 2.1 show that an approximate solution of equation (2.1) in the interval (2.2) is affected by a relatively large margin of error. For example, for t=2 the value of error is 12%:

$$y(2) = y_{śr} \pm \frac{1}{2}\frac{y-Y}{y_{śr}} \cdot 100\% = 1.8811 \pm 12\% \text{ and for t=0.2,} \quad y = 2.2479 \pm 6.3\%.$$

Tab. 2.1. Results of calculations

t [s]	**y**	**Y**	**2Δ**	**y$_{śr}$**	**y$_d$**	**β [%]**
0.10	2.819887	2.990883	0.177061	2.908418	2.93140	3.04
0.20	2.105752	2.386528	0.284454	2.247979	2.34770	6.32
0.35	1.531041	1.901105	0.371802	1.716942	1.85917	10.82
0.50	1.270942	1.683183	0.413062	1.477473	1.59496	13.97
0.65	1.164202	1.596366	0.432552	1.380478	1.45027	15.66
0.80	1.134654	1.576229	0.441758	1.355533	1.38093	16.29
1.00	1.156106	1.603007	0.446968	1.379590	1.36900	16.19
1.20	1.215397	1.664256	0.448885	1.439839	1.42420	15.58
1.50	1.347650	1.797396	0.449751	1.572526	1.59528	14.30
2.00	1.656135	2.106115	0.449980	1.881125	2.01618	11.96
3.00	2.576001	3.026001	0.450000	2.801001	3.00011	8.03
4.00	3.896000	4.346000	0.450000	4.121000	4.00000	5.45

In order to obtain greater accuracy of the solution interval (2.2) is divided into three subintervals:

$$y_1 \in < 4, .2 >, \qquad y_2 \in < 2, 1.2 >, \qquad y_3 \in < 2, .4 >. \tag{2.10}$$

In these ranges secant equation (1.3) and tangential (1.6) are the following:

$$Sc_1 = 6y - 8\,, \qquad Sc_2 = 3.2y - 2{,}4\,, \qquad Sc_3 = 6y - 8, \tag{2.11}$$

$$St_1 = 6y - 9\,, \qquad St_2 = 3.2y - 2.56\,, \qquad St_3 = 6y - 9. \tag{2.12}$$

Differential equations (2.5) and (2.6) for the interval $y_1 \in < 4.2 >$ take the form of:

$$\dot{y}_1 = t^2 + 1 - (6y_1 - 8), \tag{2.13}$$

$$\dot{Y}_1 = t^2 + 1 - (6Y_1 - 9). \tag{2.14}$$

They have the following solutions:

$$y_1 = d_1 e^{-6t} + \frac{1}{6} t\left(t - \frac{2}{6}\right) + \frac{2}{6^3} + \frac{1}{6} + \frac{8}{6}, \tag{2.15}$$

where y(t=0)=4 and d_1=2.4907;

$$Y_1 = D_1 e^{-6t} + \frac{1}{6} t\left(t - \frac{2}{6}\right) + \frac{2}{6^3} + \frac{1}{6} + \frac{9}{6}, \tag{2.16}$$

where D_1=2.3241

From equation (1.17) for the range $2 \le y_1 \le 4$ estimate error solutions:

$$2\Delta_1 = y_1 - Y_1 = \frac{-8+9}{6}(e^{-6t} - 1) \le 0.1337. \tag{2.17}$$

For the whole range $1 \le y \le 4$ error was:

$$2\Delta = Y - y = \frac{-4+6.25}{5}(1 - e^{-5t}) \le 0.45.$$

Table 2.2 shows the values $\mathbf{y_1}$ according to formula (2.15), $\mathbf{Y_1}$ by (2.16), $\mathbf{2\Delta_1}$ according to (2.17), $\mathbf{y_{1śr}}$ = $y_1 + \Delta_1$, the exact solution $\mathbf{y_d}$ according to formula (2.9) and the value of $\beta = \frac{\Delta}{y_{śr}} \cdot 100\%$.

Tab. 2.2. Values according to formula (2.15)

t [s]	y_1	Y_1	$2\Delta_1$	$y_{1śr}$	y_d	β_1 [%]
0.05	3.352054	3.395300	0.043197	3.373653	3.37547	0.64
0.10	2.872296	2.947530	0.075198	2.909895	2.93140	1.29
0.15	2.517319	2.616251	0.098905	2.566772	2.60123	1.92
0.20	2.254999	2.371487	0.116468	2.313233	2.34770	2.51
0.25	2.061537	2.191031	0.129478	2.126276	2.14823	3.04
0.269766	2.000001	2.133651	0.133637	2.066819	2.08098	3.23
0.328201	1.856600	2.000014	0.143405	1.928302	1.91236	3.71

Differential equations for the other two sub-interval $2 \geq y_2 \geq 1.2$ of the form:

$$\dot{y}_2 = t^2 + 1 - (3.2y_2 - 2.4), \tag{2.18}$$

$$\dot{Y}_2 = t^2 + 1 - (3.2Y_2 - 2.56), \tag{2.19}$$

have solutions:

$$y_2 = d_2 e^{-3.2t} + \frac{1}{3{,}2} t\left(t - \frac{2}{3.2}\right) + \frac{2}{3.2^3} + \frac{1}{3.2} + \frac{2.4}{3.2}, \tag{2.20}$$

$$Y_2 = D_2 e^{-3.2t} + \frac{1}{3{,}2} t\left(t - \frac{2}{3{,}2}\right) + \frac{2}{3.2^3} + \frac{1}{3.2} + \frac{2.56}{3.2}. \tag{2.21}$$

Table 2.2 is given final condition $y_1(t_{1k}=0.269766)=2.0000$, which is also the initial condition for the second sub-interval, i.e.:

$$y_{1k}(t_{1k} = 0.269766) = 2.0000 = y_{2p}(t_{2p} = 0.269766). \tag{2.22}$$

However $Y_1(t_{1k} = 0.269766) = 2.13365$ is not a condition for the end of the first sub-interval and the beginning of another. Here is the formula (2.14) determine the value of $t_{1k} = t_{2p}$, for which $Y_{1k} = Y_{1p} = 2.0000$. This amounts to $t_{1k}=t_{2p} = 0.3282$. Now you can set another fixed differential equations:

$$d_2(t_{1k} = 0.269766) = \left[2 - \frac{1}{3.2} t\left(t - \frac{2}{3.2}\right) - \frac{2}{3.2^3} - \frac{1}{3.2} - \frac{2.4}{3.2}\right] e^{3.2t} = 2.148972, \tag{2.23}$$

$$D_2(t_{1k} = 0.328201) = \left[2 - \frac{1}{3.2} t\left(t - \frac{2}{3.2}\right) - \frac{2}{3.2^3} - \frac{1}{3.2} - \frac{2.56}{3.2}\right] e^{3.2t} = 2.449332, \tag{2.24}$$

$$2\Delta_2 = y_2 - Y_2 = \frac{-2.4+2.56}{3.2}\left(1 - e^{3.2(t_0 - t}\right). \tag{2.25}$$

The calculations for the sub-interval $2 \geq y_2 \geq 1.2$ are shown in Tab. 2.3.

Differential equations (1.10) and (1.11) for the range $2 \leq y_3 \leq 4$ have the same form as (2.13) and (2.14) but their solutions are different from the solutions (2.15) and (2.16) the constant value of **d** i **D**:

$$y_3 = d_3 e^{-6t} + \frac{1}{6} t\left(t - \frac{2}{6}\right) + \frac{2}{6^3} + \frac{1}{6} + \frac{8}{6}, \tag{2.26}$$

$$d_3(t_{2k} = 2.01291) = \left[2 - \frac{1}{6} t\left(t - \frac{2}{6}\right) - \frac{2}{6^3} - \frac{1}{6} - \frac{8}{6}\right] e^{6t} = -12791, \tag{2.27}$$

$$Y_3 = D_3 e^{-6t} + \frac{1}{6} t\left(t - \frac{2}{6}\right) + \frac{2}{6^3} + \frac{1}{6} + \frac{9}{6}, \tag{2.28}$$

$$D_2(t_{2k} = 1.964085) = \left[2 - \frac{1}{6} t\left(t - \frac{2}{6}\right) - \frac{2}{6^3} - \frac{1}{6} - \frac{9}{6}\right] e^{6t} = -27520, \tag{2.29}$$

$$2\Delta_2 = y_3 - Y_3 = \frac{-8+9}{6}\left(1 - e^{6(t_0 - t}\right). \tag{2.30}$$

Tab. 2.3. Calculations for the sub-interval $2 \geq y_2 \geq 1.2$

t [s]	**y_2**	**Y_2**	**$2\Delta_2$**	**$y_{2śr}$**	**y_d**	**β_2 [%]**
0.269766	2.000001	2.176689	-	-	2.08098	-
0.328201	1.844918	2.000000	0.008527	1.849182	1.91236	0.23
0.35	1.794623	1.942625	0.011322	1.800284	1.85917	0.31
0.50	1.537874	1.648516	0.026067	1.550907	1.59496	0.84
0.65	1.397085	1.484609	0.035190	1.414680	1.45027	1.24
0.80	1.333411	1.406630	0.040836	1.353829	1.38093	1.50
1.00	1.328319	1.390563	0.045168	1.350903	1.36900	1.67
1.20	1.385349	1.441805	0.047452	1.409075	1.42420	1.68
1.50	1.551377	1.603849	0.049024	1.575889	1.59528	1.55
1.80	1.791244	1.842191	0.049626	1.816058	1.83476	1.36
1.964085	1.949440	2.000000	0.049779	1.974329	1.98275	1.26
2.012910	2.000004	2.050483	-	-	2.02827	-

The calculations for the sub-interval $2 \leq y_3 \leq 4$ are shown in Tab. 2.4.

Tab. 2.4. Calculations for the sub-interval $2 \leq y_3 \leq 4$

t [s]	**y_3**	**Y_3**	**$2\Delta_3$**	**$y_{3śr}$**	**y_d**	**β_3 [%]**
1.964085	1.945593	2.000000	-	-	1.98275	-
2.012910	2.000000	2.082914	0.042323	2.021162	2.02827	1.04
2.50	2.408124	2.570285	0.159977	2.488113	2.50170	3.21
3.00	2.842398	3.008840	0.166334	2.925565	3.00011	2.84
3.50	3.356472	3.523127	0.166650	3.439797	3.50000	2.42
4.00	3.953703	4.120369	0.166666	4.037036	4.00000	2.06

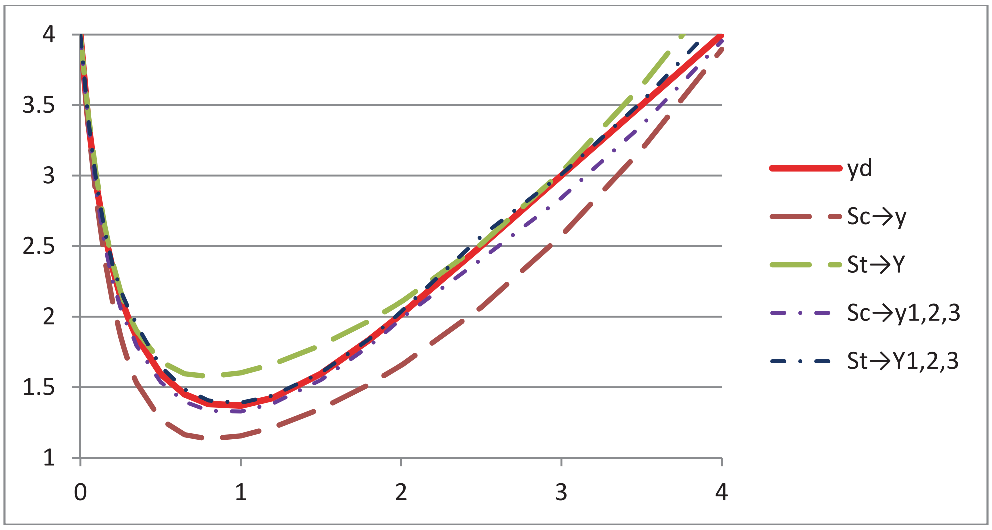

Fig. 2.1. Course exact solution y_d of the equation (2.1), approximate solution y i Y for the interval $1 \leq y \leq 4$ *, and y_1 i Y_1 , y_2 i Y_2 , y_3 i Y_3 for the intervals* $4 \geq y_1 \geq 2; 2 \geq y_2 \geq 1.2; 2 \leq y_3 \leq 4$

Summary

The presented examples of non-linearity parabolic solutions. This type of equation are described for example, in a linear motion of the ship where the resistance of the hull is a function of non-linear. That method can be extended to the non-linearity of any exponent, which shows the changing conditions of swimming. Model ship motion can be linearized, for example, in a speed range of maneuverability and the sea. The selected intervals may be divided into sub-compartments to provide the required accuracy of the model. Linearization of the model allows the use of fuzzy logic. The shortest division of the speed is maneuverability speed and sea speed of ship.

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