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SOLUTIONS OF RECURRENCE AND SUMMATION EQUATIONS AND THEIR APPLICATIONS IN SLIDE BEARING WEAR CALCULATIONS

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Abstract

The investigations under the paper include a derivation of a non-homogenous recurrence equation of the second and higher orders with variable coefficients, whose particular solutions, with the boundary conditions set and obtained from experimental measurements, will be the sequences of the wear value of the bearing in the successive years of operation. Owing to these investigations, it will be possible to predict for example the wear values of slide bearings. Moreover, this paper presents the some particular applications of recurrence equations with regard to the calculation prognosis of micro-bearing parameters such as friction forces, friction coefficients and wear. Recurrence equations are presented in a form of difference equations where the unknown functions occur as the main terms of the sequence of inquired values. In this paper, the properties of the particular values of recurrence equations are defined. Possibility of modulation and control of mentioned problem belong to the artificial intelligence of HDD micro-bearing. Presented problem describes not continuous relations hence determines the mathematical and numerical solutions in discrete spaces. Properly in the case of continuous functions, the mentioned recurrence equations have the same meaning as differential equations. Recurrent equations for discrete function correspond to differential equations for the continuous function. In final conclusions, the application of presented theory in this paper contains the numerical solutions referring the wear values of HDD bearing system in the indicated period of operating time.

Keywords: *wear prognosis, HDD micro-bearings, recurrence non-homogeneous equation*

1. Initial information about recurrence tools in tribology

The wear prognosis of two cooperating surfaces during the operation time of various technical devices has very important meaning in contemporary technological processes [1, 2, 6, 7]. Estimation, anticipation and calculation of the wear value of cooperating surfaces after many month or exploitation years of the machine or device denotes the very important parameter of material quality.

In this paper is considered the determination of wear value $\{f_n\}$ for successive years n=1,2,3,... We assume that wear f_{n+k} of cooperating surfaces in n+k time unity i.e. month or year for k=0,1,2,… is the function G of wear values f_{n+k-1}, f_{n+k-2}, …, f_{n+2}, f_{n+1}, f_n in foregoing years n+k−1, n+k−2, …, n+2, n+1, n. To solve this problem we must know the wear values f_n in n=1,2,…,k successive time units i.e. f_1, f_2, …, f_k. Our unknown particular solution is the sequence of wear values $\{f_n\}$ i.e. for example the values of a decrease in pm of HDD micro-bearing journal diameter in successive n=1,2,3,... years [4].

Mentioned problem can be written in the following n order non linear, non-homogeneous recurrence equation with variable free term: General form of recurrence equation can be expressed in following form [3, 5]:

$$f_{n+k} = G\left(f_n, f_{n+1}, f_{n+2}, ..., f_{n+k-2}, f_{n+k-1}\right), \tag{1}$$

where G is the complex function of operands f_n, f_{n+1}, f_{n+2}, …, f_{n+k-1}. Symbol f_n is the unknown wear function. Natural number k determines the order of the recurrence equation for n=1,2,… .

If G is the non linear function at least of the one of operands: f_n, …, f_{n+k-1}, then recurrence equation (1) is non linear. If G is linear respect to the all operands: f_n, …, f_{n+k-1}, then recurrence equation (1) is linear. For example linear, k-order recurrence equation with coefficients a_k has the following homogeneous and non-homogeneous (with free term b_n) form [3, 5]:

$$a_k f_{n+k} + a_{k-1} f_{n+k-1} + a_{k-2} f_{n+k-2} + ... + a_1 f_{n+1} + a_0 f_n = 0, \tag{2}$$

$$a_k f^*_{n+k} + a_{k-1} f^*_{n+k-1} + a_{k-2} f^*_{n+k-2} + ... + a_1 f^*_{n+1} + a_0 f_n{}^* = b_n \ . \tag{3a}$$

If all coefficients a_k are constants i.e. are independent of n, then formula (2) describes the recurrence equation with constant coefficients. If whichever one of coefficients a_k depends of n, then formulae (2), (3a) denote recurrence equations with variable coefficients. If free term b_n equals zero, then formula (2) describes the homogeneous recurrence equations. If free term b_n is different of zero, then formula (3a) describes the non-homogeneous recurrence equations. Experimental coefficients:

$$a_0,\ a_1,\ ...\ ,a_{k-1},\ a_k, b_n, \tag{3b}$$

depend on the surfaces material, dynamic conditions of the movable cooperating surfaces, frequencies of existing surface vibrations, environmental conditions. This section describes the methods of the determination of the general and particular solutions in the infinite sequence form for linear k-order recurrent equations with constant coefficients.

Recurrence equations (2), (3a) will be solved, if we found the following dependence [3, 5]:

$$f_n = \Phi\left(n, s_1, s_2, s_3, ..., s_k\right), \tag{4}$$

where experimental values: $f_1 = s_1, f_2 = s_2, f_3 = s_3, ..., f_k = s_k$ denote boundary wear values of searched solution of recurrence equations (1) in places for n=1,2,3,…,k.

It is easy to see, that formula (4) i.e. solution of recurrence equation (3a) or (3b) enables to calculate the values of the wear function f_n for arbitrary n without of the necessity of determining of remaining succeeding wear values f_j for j=1,2,3,...,n−1. Linear, homogeneous, k-order recurrence equations (3a) for b_n=0 has k-linear –independent particular solutions of wear creating the following fundamental system [7]:

$$f_n^{[1]}, f_n^{[2]}, f_n^{[3]}, ..., f_n^{[k-1]}, f_n^{[k]}. \tag{5}$$

Linear homogeneous, recurrence equations has as many particular solutions, as order of the equations. Upper index in brackets presented in functions (5) indicates the number of succeeding particular solutions. Casorati determinant created for particular solutions (5) can be presented in following form [3]:

$$\det\left\| f_j^n \right\| \neq 0,\ for\ j = 1,2,...,k;\ n = 1,2,...,k. \tag{6}$$

General wear solution of the linear, homogeneous recurrence equations (2) for b_n=0 is the linear combination of k-linear independent particular solutions $f_n^{[k]}$ of recurrent equation. Hence, the general solution of wear of mentioned recurrent equation has the following form:

$$f_n = C_1 \cdot f_n^{[1]} + C_2 \cdot f_n^{[2]} + C_3 \cdot f_n^{[3]} + ... + C_{k-1} \cdot f_n^{[k-1]} + C_k \cdot f_n^{[k]}, \tag{7}$$

where n=1,2,3,… .

General solution (7) contains k arbitrary constants: C_1, C_2, C_3, …, C_{k-1}, C_k. General solution of the homogeneous recurrence equations has as many constants as the order of equations. To determine the mentioned constants we need k boundary conditions. For example, such boundary conditions are as follows:

$$\begin{aligned} &\text{for } n=1, \text{ we have condition } f_1=s_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\text{for } n=k, \text{ we have condition } f_1=s_k. \end{aligned} \tag{8}$$

Symbols s_k in practical applications denote concrete wear value of looked particular solution of homogeneous recurrence equations in time unit n=k.

Imposing condition (8) on general solution (7), we obtain the following system of k- linear non-homogeneous algebraic equations [7]:

$$\begin{aligned} &C_1 \cdot f_1^{[1]} + C_2 \cdot f_1^{[2]} + C_3 \cdot f_1^{[3]} + \ldots. + C_{k-1} \cdot f_1^{[k-1]} + C_k \cdot f_1^{[k]} = s_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &C_1 \cdot f_k^{[1]} + C_2 \cdot f_k^{[2]} + C_3 \cdot f_k^{[3]} + \ldots. + C_{k-1} \cdot f_k^{[k-1]} + C_k \cdot f_k^{[k]} = s_k. \end{aligned} \tag{9}$$

The system of equations (9) determines the following unknown constants: C_1, C_2, C_3, …, C_{k-1}, C_k. It is easy to see that the Casorati determinant is the determinant of the algebraic system (6). Because such determinant is different of zero for linear independent particular solutions, hence the system of algebraic equations (9) determines non-zero constant values as solutions. Solution of recurrence equations is based on the particular and general wear solution (7) determination as linear combination of particular wear solutions. And next if we known boundary conditions i.e. if we have experimental obtained wear values s_k, then we look to obtain concrete particular solutions. Such solution we obtain putting constants C_1, C_2, C_3, …, C_{k-1}, C_k determined from the algebraic system (9) into general solution (7).

In many tribological wear, problems [5] are occurring very often the following linear particular cases of wear recurrence equation (3a):

$$\begin{aligned} f^*_{n+4} &= a\left(f^*_{n+3} + f^*_{n+2} + f^*_{n+1} + f^*_n\right) + bD^n, \\ f^*_{n+3} &= a\left(f^*_{n+2} + f^*_{n+1} + f^*_n\right) + bD^n, \\ f^*_{n+2} &= a\left(f^*_{n+1} + f^*_n\right) + bD^n, \end{aligned} \tag{10}$$

where n=1,2,3,… ; $0<a<1$.

We have experimental dimensionless values D [1], a [1] and dimensional experimental wear values b[pm], and dimensional boundary values f_1[pm], f_2[pm], f_3[pm], f_4[pm] for the first Eq.$(10)_1$; f_1[pm], f_2[pm], f_3[pm] for the second Eq.$(10)_2$; and f_1 [pm], f_2 [pm] for third Eq.$(10)_3$ all with asterisk.

In the next sections of this paper, we are going to the presentation of the methods of solutions of linear homogeneous and non-homogeneous recurrence equations with constant coefficients. Moreover, will be imposed boundary conditions on the various general wear solutions, hence will be determined the various particular wear values. Properly in the case of the space of continuous functions, the abovementioned equations have the same meaning as differential equations. The mathematical tools of solutions of recurrence equations are formulated and demonstrated using Lemmas and Theorem.

To the best of the author's knowledge, new achievements that relate to the unified discrete forms of solutions, are convenient forms for numerical calculations and the separation of the linear independent solutions.

2. Wear function described by linear k-order homogeneous recurrent equation with constant coefficients

Now examples are presented of general and particular solutions for various types of linear, homogeneous recurrent equations and for real and complex roots of characteristic equations.

2.1. Theorem

If the linear, homogeneous, k-order recurrence equations with constant coefficients ak, ak−1, ..., a1, a0 has the following form [3, 7]:

$$a_k f_{n+k} + a_{k-1} f_{n+k-1} + ... + a_1 f_{n+1} + a_0 f_n = 0, \tag{11a}$$

where fn denotes the unknown wear function, then the general solution of this equation is as follows:

$$f_n = \sum_{s=1}^{r} \chi_s^n \left(\sum_{m=0}^{\nu_s - 1} C_{sm} \cdot n^m \right), \tag{11b}$$

where Csm − arbitrary constants, n=1.2,... . The symbol χ_s where s=1,2,3,…,r; r≤k occurring in the formulas (11a,b,c) describes the succeeding, various roots of the following characteristic equation:

$$p(\chi) \equiv a_k \chi^k + a_{k-1} \chi^{k-1} + ... + a_1 \chi + a_0 = 0, \tag{12}$$

with multiplication factor νs ordered to the roots χs whereas the sum of multiplication factors of the roots is equal to the order of recurrence equation namely:

$$\nu_1 + \nu_2 + \ldots + \nu_{r-1} + \nu_r = k. \tag{13}$$

2.2. Commentator for Theorem

Successive roots of the characteristic equations (12) with multiplication factor indicated by the arrows are as follows [3]:

$$\chi_1 \to \nu_1, \chi_2 \to \nu_2, ..., \chi_r \to \nu_r. \tag{14}$$

2.3. Wear process example

Determine the wear formula of the micro-bearing journal if wear process is described by the recurrence equation:

$$f_{n+3} - f_{n+2} - f_{n+1} + f_n = 0, \quad for \quad n = 1,2,3,.... \tag{15}$$

In first succeeding three years of exploitation the values of a decrease in nm of HDD microbearing journal diameter equal to following wear values as boundary conditions:

$$f_1 = M \text{ nm}, f_2 = N \text{ nm}, f_3 = P \text{ nm}, P > M. \tag{16}$$

Solution of example

For Eq.(15), characteristic equations (12) have the following form and solutions:

$$\chi^3 - \chi^2 - \chi + 1 = 0, \ \chi_1 = 1, \quad \chi_2 = -1, \quad \chi_3 = 1. \tag{17}$$

We have $\chi_1 = 1,$ as double root ν1=2, and $\chi_2 = -1,$ as single root ν2=1. Hence we have only two various roots, thus r=2. General solution (11b) of the recurrence equations (15) is as follows:

$$f_n = (+1)^n \sum_{m=0}^{2-1} C_{1m} \cdot n^m + (-1)^n \sum_{m=0}^{1-1} C_{2m} \cdot n^m = C_{10}. + C_{11} n + C_{20} (-1)^n. \tag{18}$$

Formula (18) presents the general solution of recurrence equation (15) whereas C11, C10, C20 are the arbitrary constants and n=1,2,3,… . Now we go to the determination of the concrete particular wear solution of the recurrence equation (15) satisfying boundary conditions (16). To determine such particular solution, we impost boundary conditions (16) on the general solution (18). And we get the following values:

$$C_{10}=1.25M+0.50N-0.75P;\ C_{11}=0.50P-0.50M;\ C_{20}=-0.25M+0.50N-0.25P.$$

Inserting such values into general solution (18), we obtain the concrete particular solution of recurrence equation (15) i.e. following wear values function:

$$f_n = 1.25M + 0.50N - 0.75P + (0.50P - 0.50M)n + (-0.25M + 0.50N - 0.25P)(-1)^n. \quad (19)$$

in succeeding years n=1,2,3,… . For example in tenth years of exploitation, the wear attains value:

$$f_{10} = +N + 4{\cdot}0(P - M), \quad P > M. \quad (20)$$

Now we assume the particular case, where characteristic equations of k-degree (12) for k-order recurrence equations (11a) have R=r/2 pairs mutually conjugated complex roots with multiplication factor denoted by the symbols in succession ν1, ν2, ..., νR. The mentioned pairs we arrange in the following form [3,7]:

$$\begin{array}{lllll} (\chi_1, \overline{\chi_1}) & multiplication\ \ factor & \nu_1, & \mathrm{mod}\,\chi_1 \equiv |\chi_1|, & Arg\chi_1 \equiv \phi_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ (\chi_R, \overline{\chi_R}) & multiplication\ \ factor & \nu_R, & \mathrm{mod}\,\chi_R \equiv |\chi_R|, & Arg\chi_R \equiv \phi_R. \end{array} \quad (21)$$

2.4. Corollary

If characteristic equation (12) has not real roots and assuming only the complex roots (21), then the general solution of recurrence equations tends to the following form [3]:

$$f_n = \sum_{s=1}^{R} |\chi_s|^n \left\{ \sum_{m=0}^{\nu_s - 1} \left[C_{sm}^{(1)} \cos(n\phi_s) + C_{sm}^{(2)} \sin(n\phi_s) \right] \cdot n^m \right\}, \quad (22)$$

for s=1,2,…, R; m=0,1,2,…,νs–1 ; n=1,2,3,… where $C_{sm}^{(1)},\ C_{sm}^{(2)}$ – arbitrary constants.

2.5. Remarks

1. Because general solution (22) valid only for R pair of complex mutually conjugated roots of characteristic equations, where each pair has multiplication factor with following succession $\nu_1, \nu_2, ..., \nu_R$, and characteristic equation (12) has not real roots, then:

$$2(\nu_1 + \nu_2 + ... + \nu_{R-1} + \nu_R) = k. \quad (23)$$

   In this case, symbol k denotes even order of recurrence equation.
2. It is worth notice, that if all roots of characteristic equations are real numbers i.e. if for each s the amplitudes of a complex number ϕ_s are equal zero, then general solution (22) tends to the form of general solution (11b), whereas R=r.

3. Wear function described by linear k-order non-homogeneous recurrent equation

Conclusion

The linear, non-homogeneous, k-order recurrence equation with constant coefficients a_k and variable exponential free term [3, 5]:

$$\sum_{j=0}^{k} a_j f_{n+j}^* = b \cdot D^n, \quad a,b \neq 0, \tag{24}$$

has the following general solution:

$$f_n^* = f_n + f_n^b, \quad for \quad f_n^b \equiv \frac{D^{n-q} \cdot b \cdot n^q}{q! \cdot Q(\chi = D)}, \quad n = 1,2,3,..., \tag{25}$$

where function f_n denotes a general solution of the homogeneous, k-order recurrence equation created from the non-homogeneous Eq. (24) for b=0, and moreover f_n^b presents a certain particular solution of the non-homogeneous equation, whereas number q is the multiplicity of the root χ=D of characteristic equation p(χ)=0 see Eq.(12). The associated polynomial is defined in following form [3, 7]:

$$Q(\chi) \equiv \frac{p(\chi)}{(\chi - D)^q}. \tag{26}$$

4. Application in tribology

4.1. Problem

The sequence of wear values {fn} i.e. the values of a decrease in pm of HDD microbearing journal diameter equal to sum of wear in two foregoing successive months multiplied by dimensionless average coefficient 0<a≤1 plus some exponent dimensional function bDn. Coefficients D,a,b depend on the microbearing material, the journal angular velocity and the frequencies of vibrations. In the two first months the wear attains dimensional values W1, W2 in pm. Determine the unknown analytical formula {fn} for a sequence of wear values numbered for n=1,2,3,… month if we know dimensionless values D[1], a[1] and dimensional values b[pm], W1[pm], W2[pm].

4.2. General solution of the problem

The problem is defined by the following difference equation:

$$f_{n+2}^* = a(f_{n+1}^* + f_n^*) + bD^n \quad for \quad n = 1,2,3,... . \tag{27}$$

From formulae (11b), (25) it follows that the general solution of recurrence equation (27) for two arbitrary constants C1, C2 has the following form:

$$f_n^* = C_1 \chi_1^n + C_1 \chi_2^n + f_n^b. \tag{28}$$

Characteristic equation (12) for k=2 and recurrence equation (27) has the following real roots:

$$\chi_{1,2} = \frac{a}{2} \pm \sqrt{a + \frac{a^2}{4}}, \quad for \quad 0 < a \leq 1, D_2 \leq \chi_{1,2} \leq D_1, \quad D_{1,2} = \left(1 \pm \sqrt{5}\right)/2. \tag{29}$$

By imposing the boundary conditions $f_1^* = W_1, f_2^* = W_2$ on the general solution (28), we obtain system:

$$\begin{aligned} C_1\chi_1 + C_2\chi_2 + f_1^b &= W_1, \\ C_1\chi_1^2 + C_2\chi_2^2 + f_2^b &= W_2. \end{aligned} \tag{30}$$

System of Eqs.(30) determines following unknown constants C1, C2:

$$C_1 = -\frac{W_2 - \chi_2 W_1}{\chi_1(\chi_2 - \chi_1)} + \frac{f_2^b - \chi_2 f_1^b}{\chi_1(\chi_2 - \chi_1)}, \quad C_2 = \frac{W_2 - \chi_1 W_1}{\chi_2(\chi_2 - \chi_1)} - \frac{f_2^b - \chi_1 f_1^b}{\chi_2(\chi_2 - \chi_1)}. \tag{31}$$

4.3. Particular case of solution

We assume following assumptions:

$$\chi_1,\chi_2 \neq D, \quad \chi_{1,2} = \frac{a}{2} \pm \sqrt{a + \frac{a^2}{4}}, 0 < a < \frac{1}{2}, \ \frac{1}{2} < a \le 1 \quad i.e. \quad a \neq \frac{D^2}{D+1}, \ D_2 \le D \le D_1, \tag{32}$$

By virtue of (25), (32) the non-homogeneous solution of Eq.(27) has the form:

$$f_n^b = \frac{bD^n}{D^2 - aD - a}, \quad for \quad n = 1,2,3,... \tag{33}$$

The sum of solution (28) i.e. sum of wear values in the N time units has the following form:

$$\sum_{n=1}^{N} f_n^* = \frac{1}{\Delta\chi}\left(\chi_2 \frac{1-\chi_1^N}{1-\chi_1} - \chi_1 \frac{1-\chi_2^N}{1-\chi_2}\right) W_1 + \frac{1}{\Delta\chi}\left(\frac{1-\chi_2^N}{1-\chi_2} - \frac{1-\chi_1^N}{1-\chi_1}\right) W_2 +$$
$$- \frac{1}{\Delta\chi} \cdot \frac{bD}{D^2 - aD - a}\left[(\chi_2 - D)\frac{1-\chi_1^N}{1-\chi_1} - (\chi_1 - D)\frac{1-\chi_2^N}{1-\chi_2} - \frac{1-D^N}{1-D}\Delta\chi\right], \tag{34}$$

where $\Delta\chi \equiv \chi_2 - \chi_1$; n=1,2,3,…, N.
For additionally assumptions:

$$0 < a < \frac{1}{2}, \quad |\chi_{1,2}| < 1, \quad -\frac{1}{2} < D < 1, \tag{35}$$

then for infinitely many time units, wear (34) tends to the following limit value:

$$\sum_{n=1}^{\infty} f_n^* = \frac{1-\chi_1-\chi_2}{(1-\chi_1)\cdot(1-\chi_2)} W_1 + \frac{1}{(1-\chi_1)\cdot(1-\chi_2)} W_2 - \frac{bD}{D^2 - aD - a}\left(\frac{1+D-\chi_1-\chi_2}{(1-\chi_1)\cdot(1-\chi_2)} - \frac{1}{D-1}\right), \tag{36}$$
$$\chi_1,\chi_2 \neq D, \quad a \neq \frac{D^2}{D+1}, 0 < a < \frac{1}{2}, \quad |\chi_{1,2}| < 1, \quad -\frac{1}{2} < D < 1.$$

4.4. Particular calculation example

In two first successive time units, HDD micro-bearing journal attains diameter decreases W_1 and W_2 *nm*. Determine the wear after infinite time units using measurements and stochastic data presented by *a[1]=1/6, D[1]=1/3* for arbitrary *b [nm]*.
From formulae (29), we obtain following exploitation parameters:

$$a \neq \frac{D^2}{D+1} = \frac{1}{12}, \chi_1 = \frac{1}{2}, \chi_2 = -\frac{1}{3}, \tag{37}$$

Putting data (37) in formula (36) the wear attain the following form:

$$\sum_{n=1}^{\infty} f_n^* = \frac{5}{4} W_1 + \frac{3}{2} W_2 + \frac{3}{4} b. \tag{38}$$

5. Conclusions

A) The presented example describes non-continuous relations hence it determines the mathematical and numerical solutions in discrete spaces. Recurrent equations for the discrete function correspond to differential equations for the continuous function.
B) The application of the presented theory in mentioned examples contains the numerical solutions referring the wear values of HDD bearing system in the considered period of the operating time.
C) Unified algorithm presented constitutes a useful tool for the solution of recurrence equations applied for the wear prognosis of HDD micro-bearings [5, 7].

References

[1] Kącki, E., *Partial Differential Equations in Physical and Technical Problems,* WNT, Warsaw 1989.
[2] Kosma, Z., *Numerical Methods In Engineering Applications*, Politechnika Radomska, 1999.
[3] Koźniewska, I., *Recurrence Equations,* PWN, Warszawa 1973.
[4] Nowicki, A., *Recurrence Sequences,* Wydawnictwo Wyższej Olsztyńskiej Szkoły Informatyki i Zarządzania, Olsztyn-Toruń 2010.
[5] Wierzcholski, K., *Summation equation tools for slide microbearing systems wear prognosis*, Journal of KONES Powertrain and Transport, Vol. 18, No. 3, pp. 463-470, Warsaw 2011.
[6] Wierzcholski, K., *Mega algorithm for partial recurrence Reynolds equation*, XIV International Conference System Modelling and Control, pp. 1-8.
[7] Wierzcholski, K., *Summation equations and their applications in mechanics*, XIV International Conference System Modelling and Control, pp. 11-22.