ISSN: 1231-4005
e-ISSN: 2354-0133
DOI: 10.5604/12314005.1137474

THE LIPSCHITZ PICARD SUCCESSIVE STEP METHOD IN HYDRODYNAMIC LUBRICATION PROBLEM

Krzysztof Wierzcholski

Technical University of Koszalin, Institute of Mechatronics
Nanotechnology and Vacuum Technique
Śniadeckich Street 2, 75-453 Koszalin, Poland
tel.:+48 94 3478344, fax: +48 94 3426753
e-mail: krzysztof.wierzcholski@wp.pl

Abstract

The paper shows the successive steps of approximation of Picard unification for the solution of the non-isothermal fluid flow in thin layer including inertia forces and apparent viscosities described by the non-linear dependences. In this paper is presented a unified semi analytical method of solution of the asymmetrical, laminar, steady and unsteady, non-Newtonian lubrication problem flow between two non-rotational in general, convex, differentiable and movable surfaces when the time t depended gap between mentioned surfaces has quite an arbitrary geometry. The presented considerations relate not only to the rotational cooperating surfaces but also to the arbitrary non-rotational surfaces in general. The parallel and longitudinal intersections of mentioned surfaces are curvilinear and non-monotone in general. We consider the non-Newtonian lubricant for non-linear constitutive equations taking into account Reiner Rivlin power law relationship as well Rivlin-Ericksen formula for viscoelastic fluids.

The non-Newtonian properties create non-linear dependencies between strain and stress. Moreover, the dynamic viscosity or apparent dynamic viscosity of numerous lubricant liquids with various additions often decreases along with shear rate increasing during motion. Dynamic viscosity of lubricant fluids inside very thin micro and nano boundary layers depends on Young's modulus of the cell of surface body being in contact with the fluid.

Keywords: *Unified Pickard method, approximation of solutions, system of partial differential equations*

1. Introduction and General Basic Equations

The problem Pickard method of solution of lubrication problem had been considered already in Authors papers [7, 9]. In mentioned considerations, the computational model had been not accommodated to the curvilinear coordinates in non-isothermal flow and had been not coupled with the unified calculation algorithm. In contrary to the foregoing papers [7, 9] the presented paper utilizes a new unified Pickard calculation algorithm not only for Reiner-Rivlin power law of non-Newtonian lubricant but also for Rivlin-Ericksen viscoelastic oil properties. Such algorithm satisfies stability conditions of numerical solutions of partial differential equations and gives real values of fluid velocity components and carrying capacities occurring in journal bearing.

The Picard-unification of semi analytical method of solutions of non-Newtonian lubricant flow in thin layer gap between two cooperating surfaces is related to Reiner Rivlin power law relationship as well Rivlin-Ericksen formula for visco-elastic fluids. The analysis of the flow for the viscous fluid flow will be performed by means of the following basic equations [1, 3]:

– equation of continuity:

$$\partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v}) = 0, \tag{1}$$

– equation of motion:

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\left(\mathrm{grad}\frac{1}{2}\mathbf{v}\mathbf{v} - \mathbf{v}\times\mathrm{rot}\mathbf{v}\right) = \mathrm{Div}\mathbf{S}, \tag{2}$$

– equation of energy:

$$\operatorname{div}(\kappa \text{ grad T}) + \operatorname{div}(\mathbf{vS}) - \mathbf{v}\operatorname{Div}(\mathbf{S}) = \rho \frac{d}{dt}(c_V T). \tag{3}$$

where:
t – time,
v – lubricant fluid velocity vector with components v_i,
ρ – fluid density,
T – temperature,
c_v – fluid specific heat,
κ – fluid thermal conductivity,
S – stress tensor,
div – vector divergent,
Div – tensor divergent.

The fluid density ρ and apparent fluid viscosity η_p are variable in (α_1, α_2,α_3) directions and depend on pressure, temperature and flow shear ratio. The inertia forces are taken into account.

2. Unification attempt of constitutive dependencies

The relationship between stress tensor **S** and displacement velocity tensor $\mathbf{T_d}=\mathbf{A_1}$ i.e. constitutive equations are as follows [3]:

$$\mathbf{S} = -p\delta + \eta_p \mathbf{A_1}, \tag{4}$$

whereas unit tensor **δ**, strain tensor $\mathbf{A}_1$ have following components: δ_{ij}, Θ_{ij}. We introduce the following notations: δ_{ij} – Kronecker Delta, η_p – apparent dynamic viscosity of non-Newtonian fluid in Pas, p – pressure in Pa.

For Rivlin-Reiner fluid the apparent viscosity η_p has the following form [1, 13]:

$$\eta_{pr} = 2^{n-1} m(n) \left| \frac{1}{2}\mathbf{I}_1^2 - \mathbf{I}_2 \right|^{\frac{n-1}{2}}, \quad \mathbf{I}_1 = \Theta_{kk}, \quad \mathbf{I}_2 = \frac{1}{2} e_{ijk} e_{imn} \Theta_{jm} \Theta_{kn}, \tag{5}$$

where $\mathbf{I_1}$, $\mathbf{I_2}$ in s^{-1}, s^{-2} are the known invariants of displacement velocity tensor Θ_{ij} in s^{-1}, n – dimensionless flow index, m=m(n) – fluid consistency coefficient in Pas^n, e_{ijk} – tensor Levi-Civity.

Rivlin-Ericksen model viscoelastic properties of lubricant fluids are described by means of Rivlin-Ericksen constitutive relations. Hence their stress-strain dependencies have the following form [2, 3, 13]:

$$\mathbf{S} = -p\delta + \eta_0 \mathbf{A_1} + \alpha(\mathbf{A_1})^2 + \beta \mathbf{A_2}, \tag{6}$$

where $\mathbf{A_1}$, $\mathbf{A_2}$ velocity deformation tensors in s^{-1}, s^{-2}.

We can find such tensor **X,** which satisfies matrix equation: $\mathbf{A_2}=\mathbf{X}\cdot\mathbf{A_1}$. Hence the dependence (6) we can be written in the following approximate form:

$$\mathbf{S} = -p\delta + \mathbf{A_1}(\eta_0\delta + \alpha\mathbf{A_1} + \beta\mathbf{X}), \tag{7}$$

where the apparent viscosity η_{pe} can be written in the form [3, 11]:

$$\boldsymbol{\eta}_p \equiv \eta_o \boldsymbol{\delta} + \alpha \mathbf{A_1} + \beta\, \mathbf{X}, \;\; \eta_p = \eta_{pe} \approx \eta_o + \alpha \operatorname{tr}\mathbf{A_1} + \beta \operatorname{tr}\mathbf{X}, \tag{8}$$

where:

$$\mathbf{A_1} \equiv \mathbf{L} + \mathbf{L^T}, \; \mathbf{A}_2 \equiv \operatorname{grad} \mathbf{a} + (\operatorname{grad} \mathbf{a})^T + 2\mathbf{L^T L}, \; \mathbf{a} \equiv \mathbf{L}\, \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}, \tag{9}$$

whereas:
$\mathbf{A}_1$ – tensor of deformation of the first kind [s^{-1}],

$\mathbf{A}_2$ – tensor of deformation of the second kind [s^{-2}],
tr$\mathbf{A}_1$ – trace of tensor $\mathbf{A}_1$,
$\mathbf{L}$ – tensor of gradient of fluid velocity vector [s^{-1}],
$\mathbf{L}^T$ – transpose tensor of gradient of fluid velocity vector [s^{-1}],
$\mathbf{a}$ – acceleration vector [m/s^2],
α – first pseudo-viscosity experimental coefficient of the fluid [Pas^2],
β – second pseudo-viscosity coefficient of the fluid [Pas^2],
η_0 – dynamic viscosity of motionless fluid or for the very slow movement of fluid [Pas],
η_∞ – dynamic viscosity of fluid in large motion [Pas],
η_{pe} – apparent viscosity of liquid [Pas].

Majority of the experiments performed on the lubricant fluids indicate that dynamic viscosity decreases along with shear rate increasing [8, 10]. Hence by virtue of the obtained experimental data and using the least square methods, we can express the viscosity -shear rate relation in the following form [8, 10]:

$$\eta_{pe}(A,B) = \eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + A\cdot \mathrm{tr}(\mathbf{A_1}) + B\cdot \mathrm{tr}(\mathbf{A_1})\mathrm{tr}(\mathbf{A_1}) + B\cdot \mathrm{tr}(\mathbf{A_2})}, \tag{10}$$

where: the coefficient A, experimentally obtained, reaches values from $8\cdot10^{-6}$s to $6\cdot10^{-4}$s, and the coefficient B most often attains values from $1\cdot10^{-10}\ s^2$ to $2\cdot10^{-9}\ s^2$.

3. Thin layer boundary simplifications

The solutions are made in local curvilinear and orthogonal coordinate system (α_1, α_2, α_3) connected with the one of movable surfaces, where α_2 denotes the direction of hap height as indicated in Fig. 1a. The distance h(α_1, α_3, t) between two surfaces is significantly smaller than other dimensions of indicated surfaces. The components of the displacement vector **dr** are indicated in Fig. 1b. We have assumed that the fluid velocity components in α_1,α_3 directions have the same order of greatness [7, 9].

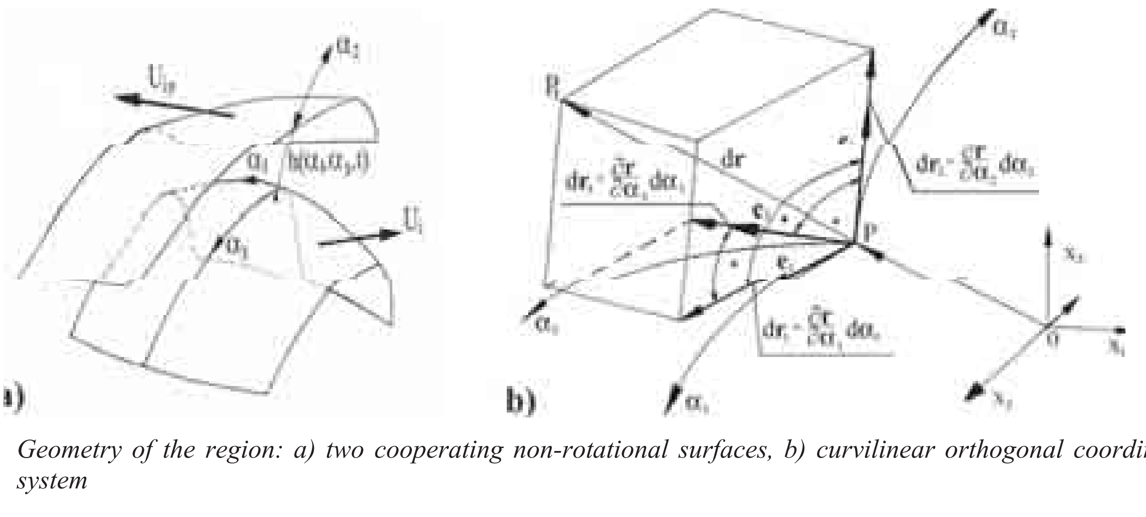

Fig. 1. Geometry of the region: a) two cooperating non-rotational surfaces, b) curvilinear orthogonal coordinates system

According to the thin boundary layer simplifications the square of the element of length in the flow region is determined as follows:

$$(\mathrm{d}\mathbf{r})^2 = h_1^2(\mathrm{d}\alpha_1)^2 + h_2^2(\mathrm{d}\alpha_2)^2 + h_3^2(\mathrm{d}\alpha_3)^2, \tag{11}$$

where:

$$h_1 \equiv h_1(\alpha_1,\alpha_3), \quad h_2 \equiv 1, \quad h_3 \equiv h_3(\alpha_1,\alpha_3) \tag{12}$$

are the Lame` coefficients in thin boundary layer depending on the shape of non-rotating surface in general. For rotational surfaces $h_1=h_1(\alpha_3)$, $h_3=h_1(\alpha_3)$. The local curvilinear and orthogonal coordinates system (α_i) connected with the lower surface is presented in Fig. 2.

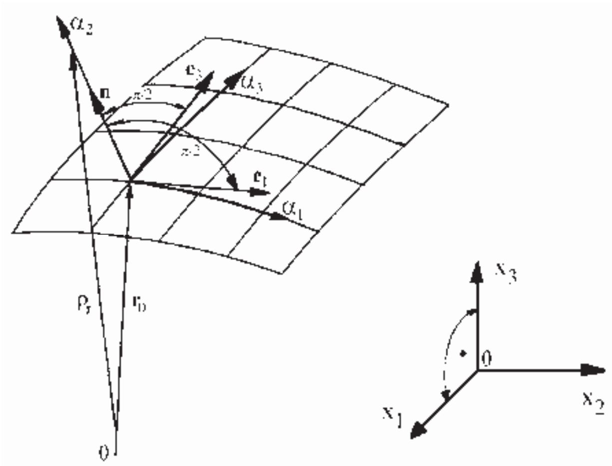

Fig. 2. Orthogonal curvilinear system connected with the surface, $\mathbf{e_i}$ *– versors in curvilinear coordinates,* **n** *– normal vector*

4. Non –Linear Basic Equations after thin layer boundary simplifications

Expanding equations (1)-(3) in α_I (i=1,2,3) directions, taking into account layer boundary simplifications, we obtain the following system of non-linear basic partial differential equations describing the lubrication of two curvilinear non-rotational surfaces [1-4]:

Equation of continuity:

$$\frac{\partial}{\partial t}(\rho h_1 h_3)+\frac{\partial}{\partial \alpha_1}(\rho v_1 h_3)+\frac{\partial}{\partial \alpha_2}(\rho v_2 h_1 h_3)+\frac{\partial}{\partial \alpha_3}(\rho v_3 h_1)=0, \tag{13}$$

Equation of motion:

For Reiner-Rivlin and Rivlin-Ericksen in equation of motion we have respectively:

$$\begin{gathered} X_i(v_1,v_3)=-\frac{1}{h_i}\frac{\partial p}{\partial \alpha_i}+\frac{\partial}{\partial \alpha_2}\left[\eta_p(v_1,v_3)\frac{\partial v_i}{\partial \alpha_2}\right], \\ 0=\frac{\partial p}{\partial \alpha_2} \quad or \quad (\alpha+2\beta)\frac{\partial}{\partial \alpha_2}\left[\left(\frac{\partial v_1}{\partial \alpha_2}\right)^2+\left(\frac{\partial v_3}{\partial \alpha_2}\right)^2\right]=\frac{\partial p}{\partial \alpha_2}. \end{gathered} \tag{14}$$

Equation of energy:

$$\begin{gathered} \frac{\partial}{\partial \alpha_2}\left(\kappa\frac{\partial T}{\partial \alpha_2}\right)+\eta_p(v_1,v_3)\left[\left(\frac{\partial v_1}{\partial \alpha_2}\right)^2+\left(\frac{\partial v_3}{\partial \alpha_2}\right)^2\right]=Z(v_1,v_3,T), \\ Z(v_1,v_3,T)\equiv\rho c_v\left(\frac{\partial T}{\partial t}+\frac{v_1}{h_1}\frac{\partial T}{\partial \alpha_1}+v_2\frac{\partial T}{\partial \alpha_1}+\frac{v_3}{h_3}\frac{\partial T}{\partial \alpha_3}\right)+\frac{p}{h_1 h_3}\left[\frac{\partial(v_1 h_3)}{\partial \alpha_1}+\frac{\partial(v_2 h_1 h_3)}{\partial \alpha_2}+\frac{\partial(v_3 h_1)}{\partial \alpha_3}\right], \end{gathered} \tag{15}$$

where for i=1,3 we have:

$$X_i(v_1,v_3)\equiv\rho\left(\frac{\partial v_i}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_i}{\partial \alpha_1}+v_2\frac{\partial v_i}{\partial \alpha_2}+\frac{v_3}{h_3}\frac{\partial v_i}{\partial \alpha_3}+\frac{v_1 v_3}{h_1 h_3}\frac{\partial h_i}{\partial \alpha_{4-i}}-\frac{v_{4-i}^2}{h_1 h_3}\frac{\partial h_{4-i}}{\partial \alpha_i}\right). \tag{16}$$

For Reiner-Rivlin and Rivlin-Ericksen we have respectively: $\eta_p(v_1,v_3,n)$, or $\eta_p(v_1,v_3,\alpha,\beta)$. The unknown functions are: velocity components v_1, v_2, v_3, pressure p, temperature T.

5. Linearization of apparent viscosity and provided solutions using small parameter method

For Rivlin-Reiner power law fluid the solutions were defined in following form of uniformly convergent power series developed in terms of small parameter [6, 7]:

$$v_i = v_{i0} + \gamma_n v_{i1} + \ldots + (\gamma_n)^j v_{ij} + \ldots ,$$
$$p = p_{d0} + \gamma_n p_{d1} + \ldots + (\gamma_n)^j p_{dj} + \ldots , \tag{17}$$
$$T = T_{d0} + \gamma_n T_{d1} + \ldots + (\gamma_n)^j T_{dj} + \ldots ,$$

for i=1,2,3; j=0,1,2,… $\gamma_n = (n-1)/2$ where $0<n\leq 3/2$ thus the small parameter γ_n is less then +1/4 and greater than −1/2. In order to attain the linearization of apparent viscosity (5), after boundary simplifications we expand function in closed interval [1,n] or [n,1] for $0<n\leq 3/2$ using Taylor series in neighbourhood of point n=1 with respect to the small parameter in following form [7, 9]:

$$\eta_p(v_1, v_3, n) = \eta_{pr} = m\left|\left(\frac{\partial v_1}{\partial \alpha_2}\right)^2 + \left(\frac{\partial v_3}{\partial \alpha_2}\right)^2\right|^{\gamma_n} = \eta_0\left[1 + \gamma_n \eta_{pr1} + \ldots + (\gamma_n)^j \eta_{prj} + \ldots\right], \tag{18}$$

where: η_0 – characteristic value of classical dynamic viscosity, η_{prj} – dimensionless expansion coefficients for j=0 we have $\eta_{pr0} \equiv 1$ and dimensionless $\eta_{prj} \equiv \eta_{prj}(v_1,v_3)$ for j=1,2,... .

For Rivlin-Ericksen type of non-Newtonian fluid the solutions were defined in following form of uniformly convergent power series developed in terms of small parameter [10]:

$$v_i = v_{i0} + D \cdot v_{i1} + \ldots + D^j \cdot v_{ij} + \ldots ,$$
$$p = p_{d0} + D \cdot p_{d1} + \ldots + D^j \cdot p_{dj} + \ldots , \tag{19}$$
$$T = T_{d0} + D \cdot T_{d1} + \ldots + D^j \cdot T_{dj} + \ldots ,$$

$$D = D_\alpha \equiv \frac{\alpha\omega}{\eta_0}, \quad or \quad D = D_\beta \equiv \frac{\beta\omega}{\eta_0}, \tag{20}$$

where: i=1,2,3; j=0,1,2,… , ω – angular velocity of journal, D – Deborah number.

Now we expand viscosity function in open interval $-1<D<1$ using Taylor series in neighbourhood of point D=0 with respect to the small parameter in following form:

$$\eta_p(v_1, v_3, \alpha, \beta) = \eta_{pe} = \eta_0\left(1 + D\eta_{pe1} + \ldots + D^j \eta_{pej} + \ldots\right). \tag{21.1}$$

where: η_0 – characteristic value of classical dynamic viscosity, η_{pej} – dimensionless expansion coefficients whereas for j=0 we have $\eta_{pe0} \equiv 1$ and dimensionless $\eta_{pej} \equiv \eta_{pej}(v_1,v_3)$ for j=1,2,... [13]:

$$\eta_{pe1}(v_1, v_3) = \frac{1}{1!}\left\{\frac{\partial \eta_{pe}(v_1, v_3, D)}{\partial D}\right\}_{D=0}, \quad \eta_{pe2}(v_1, v_3) = \frac{1}{2!}\left\{\frac{\partial^2 \eta_{pe}(v_1, v_3, D)}{\partial D^2}\right\}_{D=0}, \ldots . \tag{21.2}$$

6. Solution of the system of partial basic equations as terms of series

Putting series (17), (18) or (20), (21) into the system of non-linear equation (13), (14), (15) and multiplying the series by Cauchy method, equating the coefficients of the like powers of small parameters, we obtain a sequence of following systems of non-linear (for $X_{ij}\neq 0$), or linear (for $X_{ij}=0$) partial equations [9, 12]:

$$X_{ij}(v_{i0}, v_{i1}, \ldots, v_{ij}) + \frac{1}{h_i}\frac{\partial p_{dj}}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_2}\left(\eta_0 \frac{\partial v_{ij}}{\partial \alpha_2}\right) + \frac{\partial}{\partial \alpha_2}(\eta_0 S_{ij}), \quad i = 1,3, \tag{22}$$

$$\frac{\partial p_{dj}}{\partial \alpha_2} = \zeta \frac{\partial F_j}{\partial \alpha_2}, \tag{23}$$

$$\delta_{j0} \frac{\partial}{\partial t}(\rho h_1 h_3) + \frac{\partial}{\partial \alpha_1}(\rho v_{1j} h_3) + \frac{\partial}{\partial \alpha_2}(\rho v_{2j} h_1 h_3) + \frac{\partial}{\partial \alpha_3}(\rho v_{3j} h_1) = 0, \tag{24}$$

$$\frac{\partial}{\partial \alpha_2}\left(\kappa \frac{\partial T_{dj}}{\partial \alpha_2}\right) + \eta_0 G_j = Z_j(v_{10},..,v_{ij}; T_{d0}, T_{d1},...., T_{dj}), \tag{25}$$

for i=1,3; j=0,1,2,…, $\zeta \equiv (1+2\beta/\alpha)\eta_0/\omega$ where

$$\begin{gathered} S_{i0} \equiv 0, \quad S_{i1} = \frac{\partial v_{i0}}{\partial \alpha_2}\eta_{p1}, \quad S_{i2} = \frac{\partial v_{i0}}{\partial \alpha_2}\eta_{p2} + \frac{\partial v_{i1}}{\partial \alpha_2}\eta_{p1}, ..., \ S_{ij} = S_{ij}(v_{1j-1}, v_{3j-1}), \\ G_1 = \eta_{p1}G_0 + 2G_{11}, \ G_2 = \eta_{p2}G_0 + 2\eta_{p1}G_{11} + 2G_{12} + G_{13}, ..., \\ F_0 = 0, \quad F_1 = G_0, \quad F_2 = 2G_{11}, \quad F_3 = 2G_{12} + G_{13}, ..., \ F_j = F_j(v_{1j-1}, v_{3j-1}), \\ \eta_{p1} = \eta_{p1}(v_{10}, v_{30}), \eta_{p2} = \eta_{p2}(v_{11}, v_{31}), ..., \eta_{pj} = \eta_{pj}(v_{1j-1}, v_{3j-1}), \\ G_0 \equiv \left(\frac{\partial v_{10}}{\partial \alpha_2}\right)^2 + \left(\frac{\partial v_{30}}{\partial \alpha_2}\right)^2, \ G_{11} = \frac{\partial v_{10}}{\partial \alpha_2}\frac{\partial v_{11}}{\partial \alpha_2} + \frac{\partial v_{30}}{\partial \alpha_2}\frac{\partial v_{31}}{\partial \alpha_2}, \\ G_{12} = \frac{\partial v_{10}}{\partial \alpha_2}\frac{\partial v_{12}}{\partial \alpha_2} + \frac{\partial v_{30}}{\partial \alpha_2}\frac{\partial v_{32}}{\partial \alpha_2}, \quad G_{13} = \left(\frac{\partial v_{11}}{\partial \alpha_2}\right)^2 + \left(\frac{\partial v_{31}}{\partial \alpha_2}\right)^2, ..., \end{gathered} \tag{26}$$

where $\eta_{pj} = \eta_{prj}$ or $\eta_{pj} = \eta_{pej}$.

Analogically we obtain functions: Z_0, Z_1, … . System of Eqs.(22)-(25) for $X_{ij}=0$, $Z_j=0$ determines following unknown functions: v_{1j}, v_{2j}, v_{3j}, p_{dj}, T_{dj} for i=1,3; j=0,1,2,... .

Since the two cooperating surfaces are moving, and there can be slip, hence the boundary conditions (for i=1,2,3; j=0,1,2,…) have the following form [5, 7, 9]:

$$v_{ij}(\alpha_1, \alpha_2 = 0, \alpha_3, t) = \delta_{j0} U_i(\alpha_1, \alpha_3, t), \tag{27}$$

$$v_{ij}(\alpha_1, \alpha_2 = h, \alpha_3, t) = \delta_{j0} U_{ip}(\alpha_1, \alpha_3, t), \tag{28}$$

where δ_{j0} denotes Delta Kronecker Symbol. Velocities and slips on the journal and sleeve surface $U_i \geq 0$, $U_{ip} \geq 0$ can be continuous, constant or variable but not arbitrary in general.

– We put j=0, $S_{i0}=F_0=0$ in system (22)-(25). Hence system determines basic functions:

$$v_{10},\ v_{20},\ v_{30},\ p_{d0},\ T_{d0}. \tag{29.0}$$

– We put j=1, $S_{i0}=F_0=0$, S_{11}, F_1, G_1 and solutions (29.0) in system (22)-(25). Hence system determines following correction functions:

$$v_{11},\ v_{21},\ v_{31},\ p_{d1},\ T_{d1} \text{ i.e. } Dv_{11},\ Dv_{21},\ Dv_{31},\ Dp_{d1},\ DT_{d1}. \tag{29.1}$$

– We put j=2, $S_{i0}=F_0=0$, S_{i1}, F_1, G_1, S_{i2}, F_2, G_2, and solutions (29.1) in system (22)-(25). Hence system next correction functions:

$$v_{12},\ v_{22},\ v_{32},\ p_{d2},\ T_{d2} \text{ i.e. } D^2v_{12},\ D^2v_{22},\ D^2v_{32},\ D^2p_{d2},\ D^2T_{d2}. \tag{29.2}$$

– After J steps we obtain final corrections:

$$v_{1j},\ v_{2j},\ v_{3j},\ p_{dj},\ T_{dj} \text{ i.e. } D^Jv_{1J},\ D^Jv_{2J},\ D^Jv_{3J},\ D^Jp_{dJ},\ D^JT_{dJ}. \tag{29.J}$$

7. The method of Picard successive approximation steps of solutions of basic equations

When we are neglecting the inertia forces i.e. $X_{ij}=0$ and convection transport of energy as well pressure dissipation i.e. $Z_j=0$, then linear set of partial differential equations (22)-(25) gives following solutions (29.0), (29.1), …, (29.j) namely:

$$v_{1j} \approx v_{1j}^{(0)}, v_{2j} \approx v_{2j}^{(0)}, v_{3j} \approx v_{3j}^{(0)}, p_{dj} \approx p_{dj}^{(0)}, T_{dj} \approx T_{dj}^{(0)} \text{ for j=0,1,2, …} \tag{30}$$

We put functions (30) into terms $\mathrm{X_{ij}}, \mathrm{Z_j}$ i.e.:

$$X_{ij}^{(0)} \equiv X_{ij}(v_{i0}^{(0)}, v_{i1}^{(0)}, ..., v_{ij}^{(0)}), \quad Z_j^{(0)} \equiv Z_j(v_{i0}^{(0)}, .., v_{ij}^{(0)}; T_{d0}^{(0)}, T_{d1}^{(0)}, ..., T_{dj}^{(0)}), \tag{31}$$

for i=1,3; j=0,1,2,… .

In partial differential equations (22)-(25) we are replacing the inertia forces i.e. X_{ij} and convection transport of energy as well pressure dissipation i.e. Z_j, by functions (31). Hence such linear set of partial differential equations (22)-(25) gives following solutions:

$$v_{1j} \approx v_{1j}^{(1)}, v_{2j} \approx v_{2j}^{(1)}, v_{3j} \approx v_{3j}^{(1)}, p_{dj} \approx p_{dj}^{(1)}, T_{dj} \approx T_{dj}^{(1)} \text{ for j=0,1,2,… .} \tag{32}$$

We put functions (32) into terms $\mathrm{X_{ij}}, \mathrm{Z_j}$ i.e.:

$$X_{ij}^{(1)} \equiv X_{ij}(v_{i0}^{(1)}, v_{i1}^{(1)}, ..., v_{ij}^{(1)}), \quad Z_j^{(1)} \equiv Z_j(v_{i0}^{(1)}, ..., v_{ij}^{(1)}; T_{d0}^{(1)}, T_{d1}^{(1)}, ..., T_{dj}^{(1)}), \tag{33}$$

for i=1,3; j=0,1,2,… .

In partial differential equations (22)-(25) we are replacing the inertia forces i.e. X_{ij} and convection transport of energy as well pressure dissipation i.e. Z_j, by functions (33). Hence such linear set of partial differential equations (22)-(25) gives following solutions:

$$v_{1j} \approx v_{1j}^{(2)}, v_{2j} \approx v_{2j}^{(2)}, v_{3j} \approx v_{3j}^{(2)}, p_{dj} \approx p_{dj}^{(2)}, T_{dj} \approx T_{dj}^{(2)} \text{ for j=0,1,2,… .} \tag{34}$$

After k steps inertia force terms and convection transport terms for i=1,2,3; j=0,1,2,… go to the form:

$$X_{ij}^{(k)} \equiv X_{ij}(v_{i0}^{(k)}, v_{i1}^{(k)}, ..., v_{ij}^{(k)}), \quad Z_j^{(k)} \equiv Z_j(v_{i0}^{(k)}, ..., v_{ij}^{(k)}, T_{d0}^{(k)}, T_{d1}^{(k)}, ..., T_{dj}^{(k)}), \tag{35}$$

whereas linear set of partial differential equations (22)-(25) gives following solutions:

$$v_{1j} \approx v_{1j}^{(k+1)}, v_{2j} \approx v_{2j}^{(k+1)}, v_{3j} \approx v_{3j}^{(k+1)}, p_{dj} \approx p_{dj}^{(k+1)}, T_{dj} \approx T_{dj}^{(k+1)} \text{ for j=0,1,2,… .} \tag{36}$$

If for sufficient many steps and negligibly small value θ are valid following inequalities:

$$\left|v_{ij}^{(k)} - v_{ij}^{(k+1)}\right| < \theta, \quad \left|p_{dj}^{(k)} - p_{dj}^{(k+1)}\right| < \theta, \quad \left|T_{dj}^{(k)} - T_{dj}^{(k+1)}\right| < \theta, \tag{37}$$

then for i=1,2,3; j=0,1,2,… are valid following limits

$$\lim_{k\to\infty} X_{ij}^{(k)} = X_{ij}^{*}, \quad \lim_{k\to\infty} Z_i^{(k)} = Z_i^{*}, \lim_{k\to\infty} v_{ij}^{(k)} = v_{ij}, \quad \lim_{k\to\infty} p_{dj}^{(k)} = p_{dj}, \quad \lim_{k\to\infty} T_{dj}^{(k)} = T_{dj}, \tag{38}$$

i.e. solutions (36) have final form.

8. Conclusions

In this paper, the Lipschitz Picard's method of successive approximations of solutions of hydrodynamic lubrication problem is presented for the non-linear fluid mechanics equations that describe non-Newtonian fluid flow using Reiner-Rivlin and Rivlin Ericksen model describing non-linear apparent lubricant viscosity. The convergence process of the sequence of succeeding approximation solutions has been considered.

References

[1] Astarita, G., Marrucci, G., *Principles of non-Newtonian fluid mechanics*, McGraw Hill Co, 1974.

[2] Miszczak, A., *Analiza hydrodynamicznego smarowania ferrocieczą poprzecznych łożysk ślizgowych*, Fundacja Rozwoju Akademii Morskiej w Gdyni, dysertacja habilitacyjna, 2006.

[3] Truesdell, C. A., *First Course in Rational Continuum Mechanics,* John Hopkins Univ., Baltimore Maryland 1972.
[4] Walicki, E., *Ruch płynów lepkich w szczelinach wzdłużnych łożysk ślizgowych*, Wydawnictwo Uczelniane ATR Bydgoszcz, Mechanika, Z. 18 (50), Bydgoszcz 1977.
[5] Weinberger, H., Fo., *A First Course In Partial Differentia Equations*, John Willey & Sons, New York, Toronto, Singapore 1965.
[6] Wierzcholski, K., Czajkowski, A., *On the small parameter methods for the integration of non linear system of equations of the laminar non-Newtonian fluid flow in the thin layer on the rotating surface*, Folia Scientis Scientiarum Lubliniensis, Mat-Fiz-Chem, Vol. 31, 1-2, pp. 9-16, 1989.
[7] Wierzcholski, K., *The Method of Solving of the System of non-Linear Differential Equations for non- Isothermic Laminar non-Newtonian Flow in the Thin Layer Between Two Certain Surfaces,* Scientific Papers of the Institute of Machine Design and Operation Studies and Research: The boundary integral equation method in fluid mechanics, Technical University of Wroclaw, No. 59, Ser.: No. 26, pp. 134-144, 1993.
[8] Wierzcholski, K., *The method of solutions for hydrodynamic lubrication by synovial fluid flow in human joint gap,* Control and Cybernetics, Vol. 31, No. 1, pp. 91-116, 2002.
[9] Wierzcholski, K., *Estimation of solutions of basic equations for non Newtonian fluid flow in a film between two non- rotational surfaces.* Rev. Roum. des Sci. Tech., Ser. de Mec. Appl. Editura Acad. Roum., No. 1-2, T. 36, pp.103-122, 1991.
[10] Wierzcholski, K., *Comparisson Between Impulsive and Periodic Non-Newtonian Lubrication of Human Hip Joint,* Engineering Trans., 53, 1, pp. 69-114, 2005.
[11] Wierzcholski, K., *Bio and slide bearings: their lubrication by non-Newtonian fluids and application in non conventional systems, Vol.III: Tribology process for chondrocytes,human joint and micro-bearing*, Monograph, Published by Krzysztof Wierzcholski, Gdansk University of Technology, pp. 1-129, Gdańsk 2006-2007.
[12] Wierzcholski, K., *Mathematical implementation into computer calculations for micro-bearing capacities*, XIII Journal of Applied Computer Science, Vol. 18, No. 1, pp. 117-135, 2010.
[13] Wierzcholski, K., *Computer Simulations of Short Radial Bearing Capacity for Rivlin Ericksen Oils*. SAMS, System Analysis Modelling and Simulations OPA Overseas Publishers. Association N.V., Vol. 33, pp. 219-238, 1998.